

Practical Identification of Dispersive Dielectric Models with Generalized Modal S-parameters for Analysis of Interconnects in 6-100 Gb/s Applications

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Outline

- Project goals
- Computation of generalized modal S-parameters
- Measurement of generalized modal S-parameters
- Identification of dielectric models
- Practical examples
- Conclusion

Why do we need broadband dielectric models?

- ❑ Insulators or dielectrics are the media where signals propagate along the conductors of interconnects
- ❑ PCB dielectrics exhibit strong **dependency on frequency**
- ❑ Dielectric constant and loss tangent changing substantially over the frequency band of multi-gigabit signal spectrum
 - From DC up to 20 GHz for 10-20 Gb/s
 - From DC up to 40 GHz for 20-40 Gb/s
- ❑ *Dielectric models are the requisite foundation for performing meaningful electromagnetic verification of multi-gigabit interconnects*

Why are obtaining accurate dielectric models so difficult?

- ❑ Manufacturers of dielectrics and PCBs provide measurements for dielectric constant and loss tangent typically at **one frequency point or at 2-3 points in the best cases**
- ❑ Only frequency-continuous models can describe dispersive behavior of PCB dielectrics over very wide bandwidth
- ❑ Simplified TDR-based methods and advanced microwave resonator-based methods do not produce dispersive dielectric models
- ❑ *Multi-gigabit interconnect **design and compliance analysis must start with the identification of the dielectric properties** over the frequency band of interest*

Project goals

- Develop **simple and accurate method** for dispersive dielectric parameters identification
 - Use VNA and SOLT calibration only (no TRL)
 - Suitable both for test and production boards

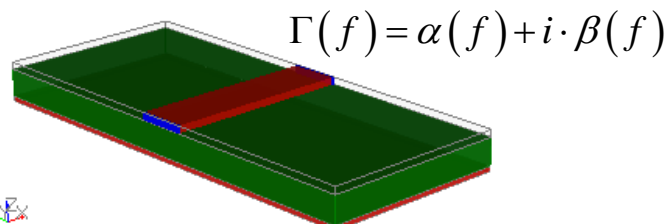
- Verify it on low-cost high-loss FR-4 and on low-loss high-performance composite dielectrics

Outline

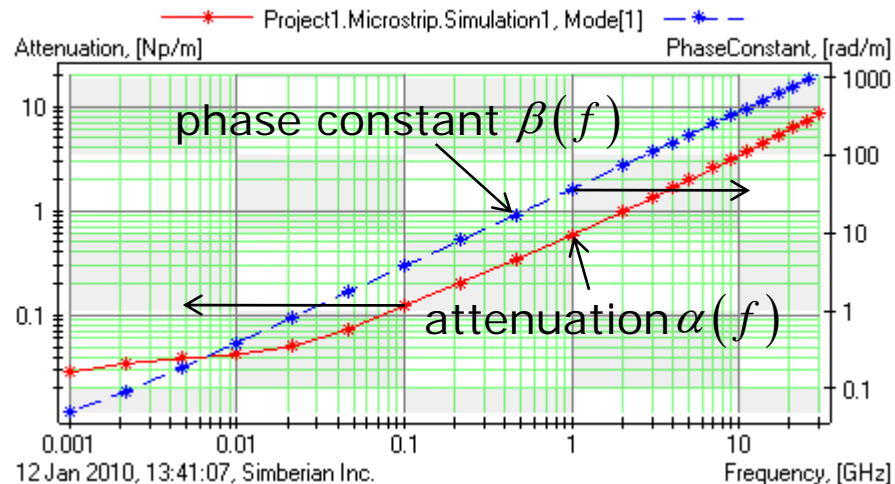
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Generalized modal S-parameters for single conductor line

1. Compute propagation constant (Gamma)



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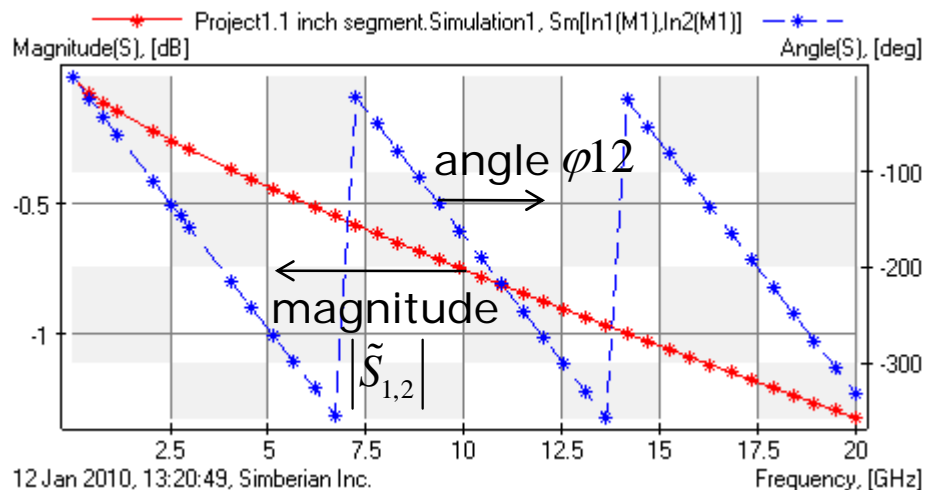


2. Compute 2x2 Sg of line segment with length l

$$\tilde{S}_g(f, l) = \begin{bmatrix} 0 & \tilde{S}_{1,2} \\ \tilde{S}_{1,2} & 0 \end{bmatrix}$$

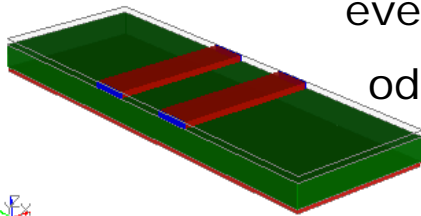
$$\tilde{S}_{1,2} = e^{-\Gamma(f) \cdot l} = |\tilde{S}_{1,2}| e^{i \cdot \varphi_{12}}$$

Very simple!



Generalized modal S-parameters for two-conductor line

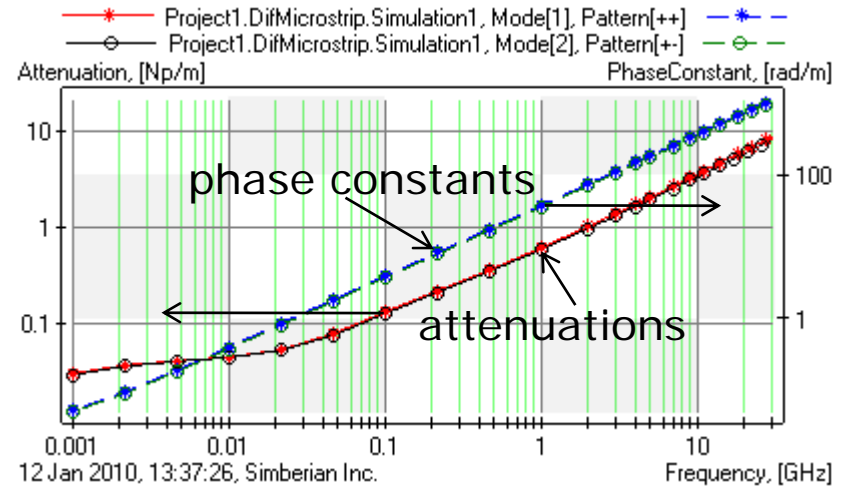
1. Compute propagation constants for 2 modes



even $\Gamma_1(f) = \alpha_1(f) + i \cdot \beta_1(f)$

odd $\Gamma_2(f) = \alpha_2(f) + i \cdot \beta_2(f)$

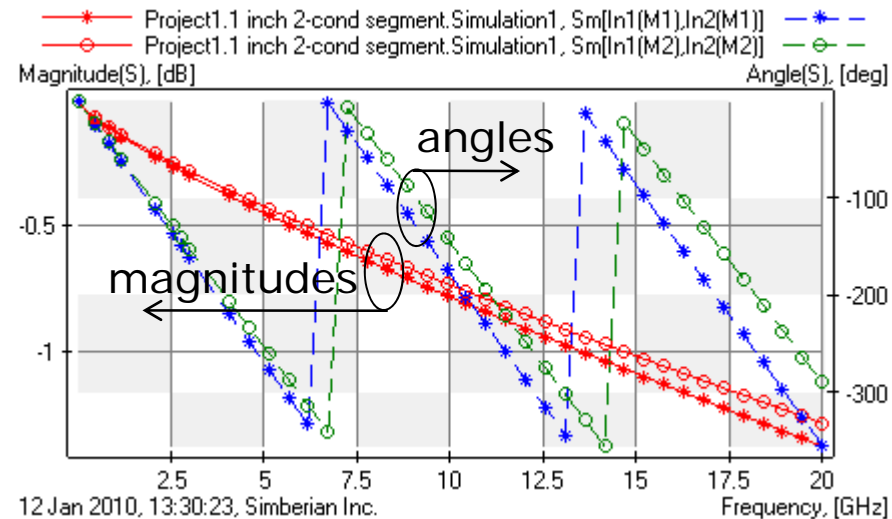
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2. Compute 4x4 Sg of line segment with length l

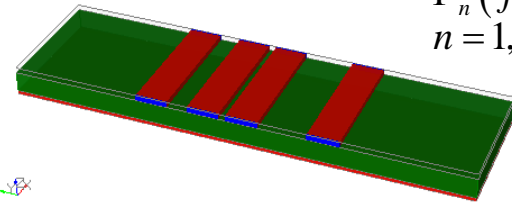
$$\tilde{S}g(f, l) = \begin{bmatrix} 0 & 0 & \tilde{S}_{1,3} & 0 \\ 0 & 0 & 0 & \tilde{S}_{2,4} \\ \tilde{S}_{1,3} & 0 & 0 & 0 \\ 0 & \tilde{S}_{2,4} & 0 & 0 \end{bmatrix} \quad \begin{aligned} \tilde{S}_{1,3} &= e^{-\Gamma_1(f) \cdot l} \\ \tilde{S}_{2,4} &= e^{-\Gamma_2(f) \cdot l} \end{aligned}$$

Simple!



Generalized modal S-parameters for N-conductor line

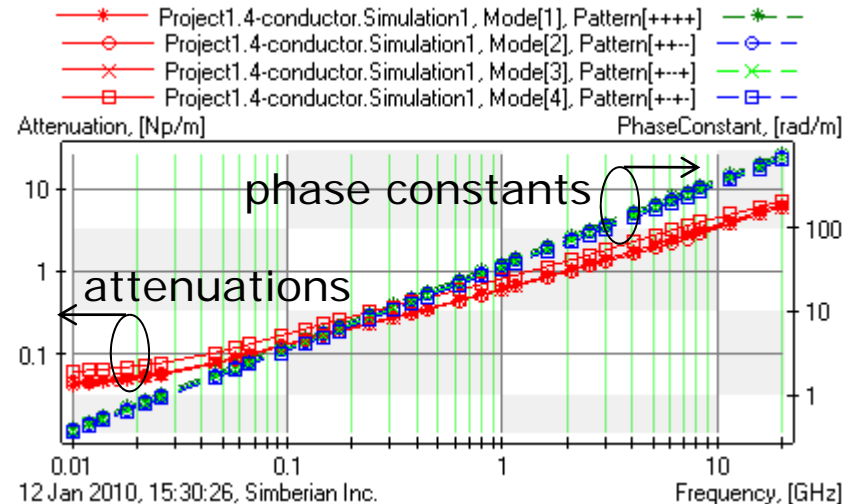
1. Compute propagation constants for N modes



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$$\Gamma_n(f) = \alpha_n(f) + i \cdot \beta_n(f), \quad n = 1, \dots, N$$

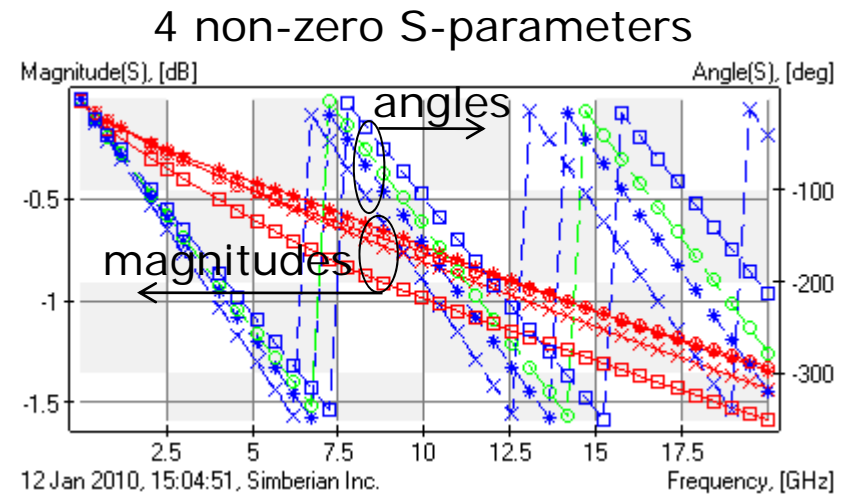


2. Compute NxN Sg of line segment with length l

$$\tilde{S}g(f, l) = \begin{bmatrix} 0 & Sm \\ Sm & 0 \end{bmatrix}$$

$$Sm = \text{diag} \left(e^{-\Gamma_n(f) \cdot l}, n = 1, \dots, N \right)$$

Simple too!

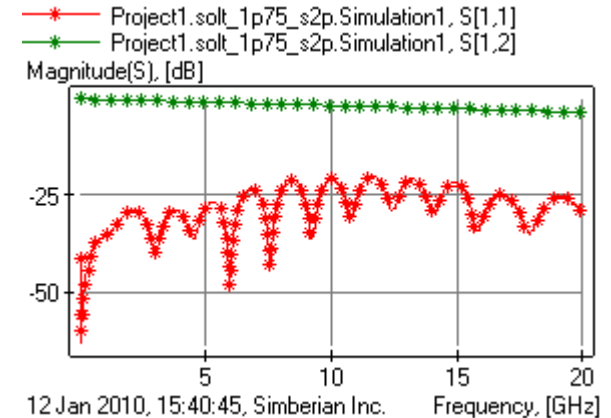
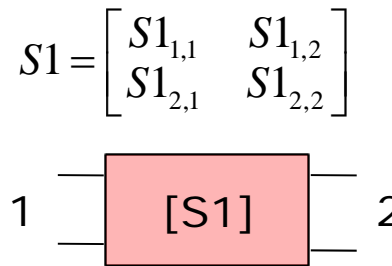
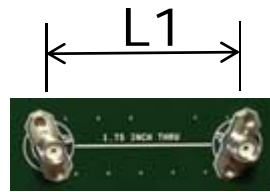


Outline

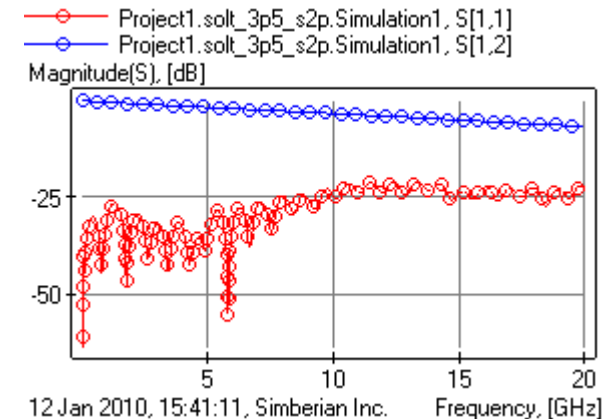
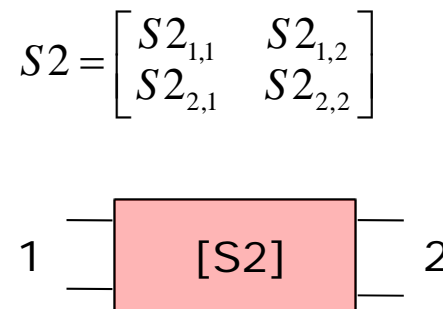
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Measure S-parameters of two line segments

□ S1 for line with length L1



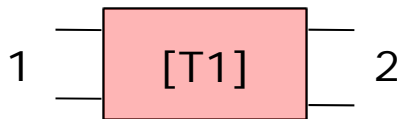
□ S2 for line with length L2



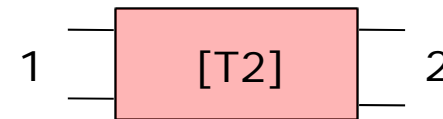
Convert S-parameters of line segments into T-parameters



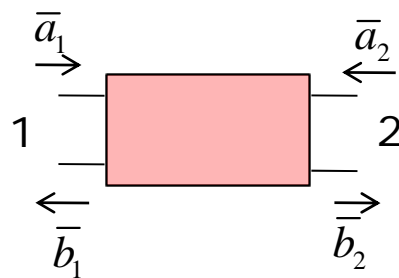
$$S1 = \begin{bmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{bmatrix} \rightarrow T1 = \begin{bmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{bmatrix}$$



$$S2 = \begin{bmatrix} S_{2,1} & S_{2,2} \\ S_{2,1} & S_{2,2} \end{bmatrix} \rightarrow T2 = \begin{bmatrix} T_{2,1} & T_{2,2} \\ T_{2,1} & T_{2,2} \end{bmatrix}$$



conversion equations



$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} = \begin{bmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{bmatrix} \cdot \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \bar{b}_1 \\ \bar{a}_1 \end{bmatrix} = \begin{bmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{bmatrix} \cdot \begin{bmatrix} \bar{a}_2 \\ \bar{b}_2 \end{bmatrix}$$

$$T_{1,1} = S_{2,1} - S_{1,1} \cdot S_{2,1}^{-1} \cdot S_{2,2}$$

$$T_{1,2} = S_{1,1} \cdot S_{2,1}^{-1}$$

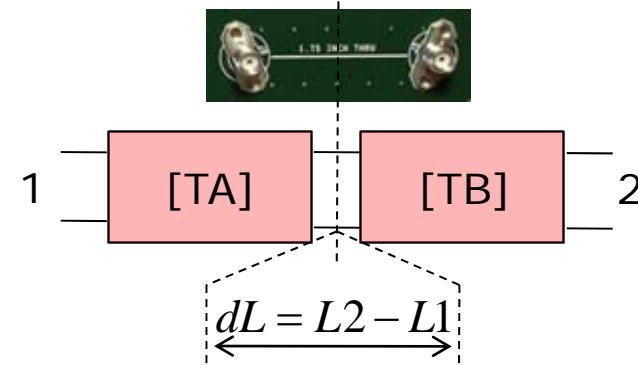
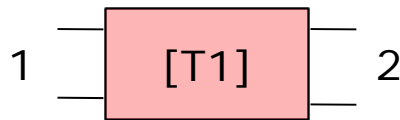
$$T_{2,1} = -S_{2,1}^{-1} \cdot S_{2,2}$$

$$T_{2,2} = S_{2,1}^{-1}$$

T-matrices decomposition and TG

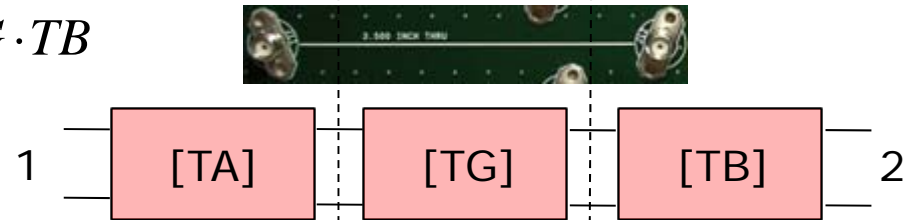
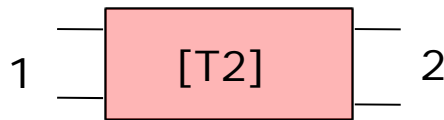
Segment L1

$$T1 = TA \cdot TB$$



Segment L2

$$T2 = TA \cdot TG \cdot TB$$



TG is non-reflective modal T-matrix (normalized to the unknown characteristic impedances of the modes)

$$T2 \cdot T1^{-1} = TA \cdot TG \cdot TA^{-1}$$



$$TG = \text{eigenvals}(T2 \cdot T1^{-1})$$

Easy to compute!

$$TG = \begin{bmatrix} e^{-\Gamma(f) \cdot dL} & 0 \\ 0 & e^{\Gamma(f) \cdot dL} \end{bmatrix} \quad \text{1-conductor line}$$

$$TG = \begin{bmatrix} Tm & 0 \\ 0 & (Tm)^{-1} \end{bmatrix} \quad Tm = \text{diag} \left(e^{-\Gamma_n(f) \cdot dL}, n = 1, \dots, N \right) \quad \text{N-conductor line}$$

Convert TG into generalized modal S-parameters of the line difference

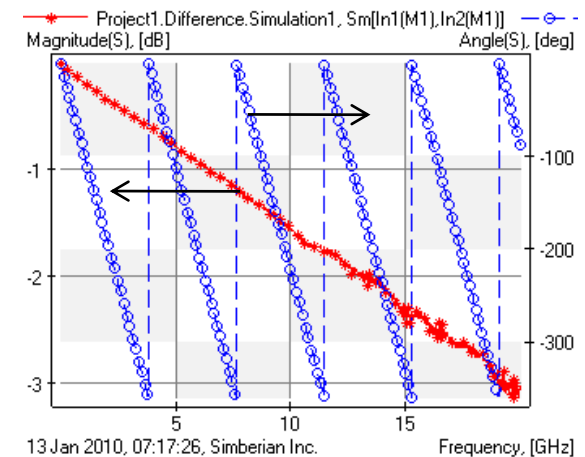
□ 1-signal conductor case

$$TG = \begin{bmatrix} e^{-\Gamma(f) \cdot dL} & 0 \\ 0 & e^{\Gamma(f) \cdot dL} \end{bmatrix} \rightarrow SG = \begin{bmatrix} 0 & e^{-\Gamma(f) \cdot dL} \\ e^{-\Gamma(f) \cdot dL} & 0 \end{bmatrix}$$

□ N-conductor case

$$TG = \begin{bmatrix} Tm & 0 \\ 0 & (Tm)^{-1} \end{bmatrix} \rightarrow SG = \begin{bmatrix} 0 & Sm \\ Sm & 0 \end{bmatrix}$$

$$Sm = Tm = \text{diag} \left(e^{-\Gamma_n(f) \cdot dL}, n = 1, \dots, N \right)$$



- Measured SG of the difference segment can be directly compared with calculated generalized modal S-parameters for dielectric parameters identification

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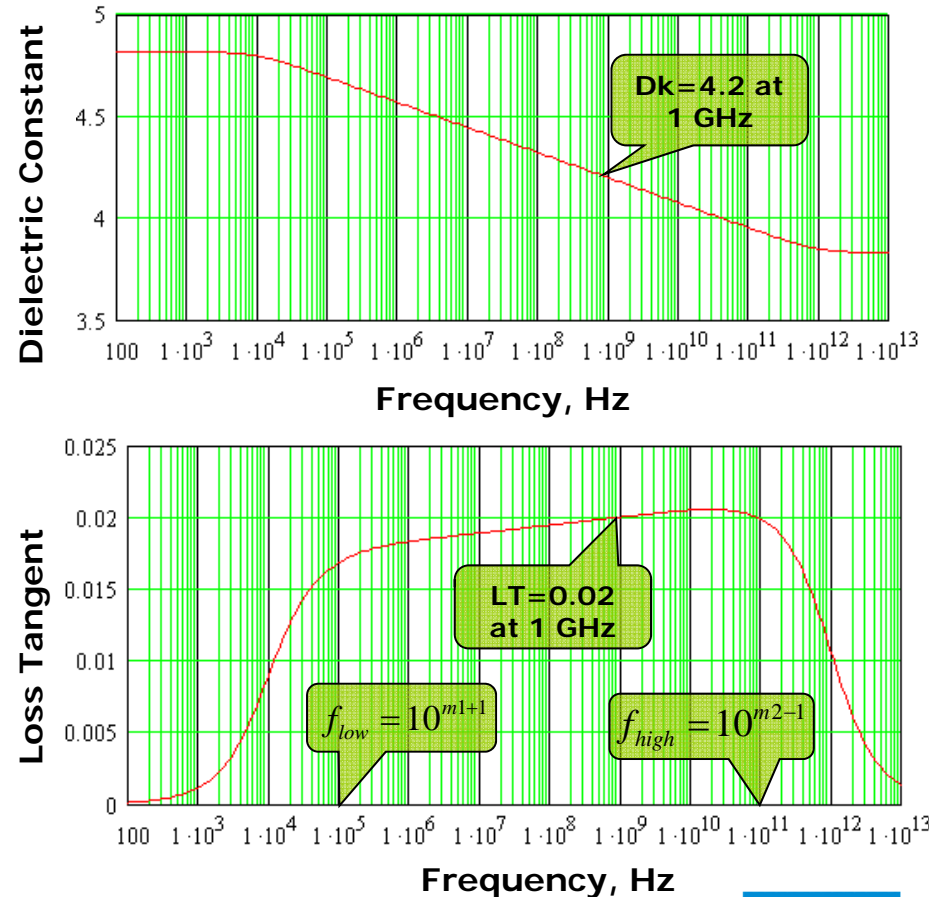
Possible continuous dielectric model: Wideband Debye model

$$\varepsilon_{wd}(f) = \varepsilon_r(\infty) + \varepsilon_{rd} \cdot F_d(f)$$

$$F_d(f) = \frac{1}{(m_2 - m_1) \cdot \ln(10)} \cdot \ln \left[\frac{10^{m_2} + if}{10^{m_1} + if} \right]$$

- Continuous-spectrum model
- Requires specification of DK and LT at one frequency point
- Good match for high-loss FR-4 dielectrics (LT>0.01)
- Unfortunately does not provide good match for low-loss, high-frequency composites (LT<0.01)

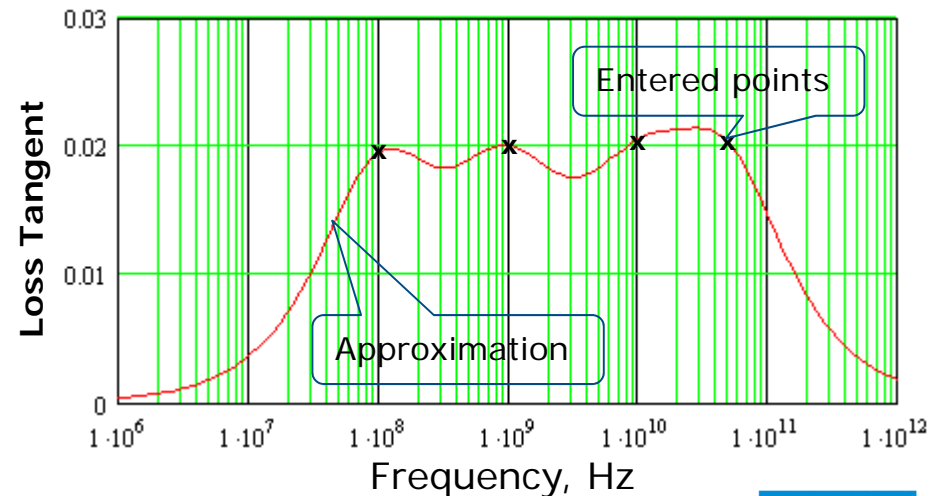
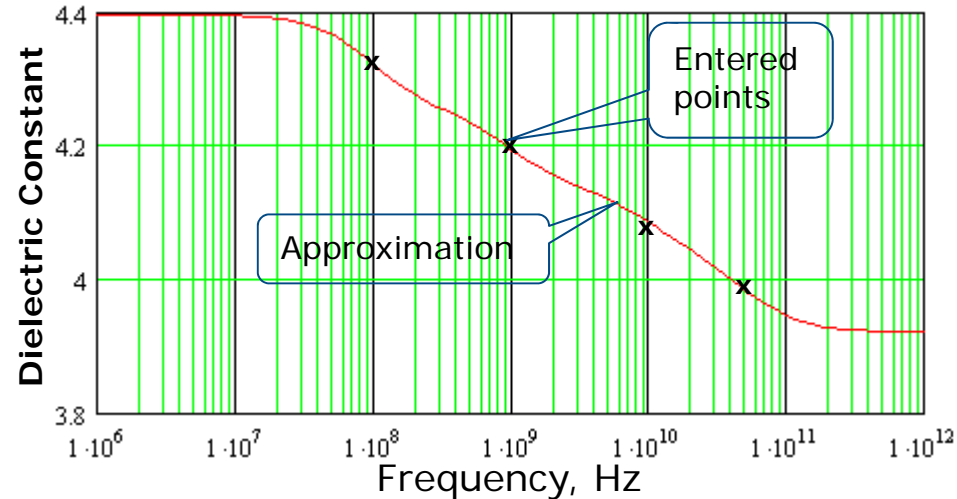
Djordjevic, R.M. Biljic, V.D. Likar-Smiljanic, T.K.Sarkar, IEEE Trans. on EMC, vol. 43, N4, 2001, p. 662-667.



Possible continuous dielectric model: Multi-pole Debye model

$$\varepsilon(f) = \varepsilon(\infty) + \sum_{n=1}^N \frac{\Delta\varepsilon_n}{1 + i \frac{f}{fr_n}}$$

- Discrete-spectrum model
- Requires specification of DK and LT at multiple frequency points
- Can be used for any dielectric without resonances
- At least 4 poles (usually 10) are required for composite dielectrics for multi-gigabit signals



Dielectric model identification procedure

- ❑ Measure S-parameters for two line segments S1 and S2
- ❑ Transform S1 and S2 to the T-matrices T1 and T2, diagonalize the product of T1 and inversed T2 and compute generalized modal S-parameters of the line difference SG
- ❑ Select dielectric model and guess values of the model parameters
- ❑ Compute generalize modal S-parameters Sg of the line difference segment by solving Maxwell's equation for t-line cross-section (only propagation constants are needed)
- ❑ Compare simulated Sg and measured SG modal transmission parameters and adjust dielectric model until simulated data match the measured data

Simberian's patent pending (application #61296237)

The technique is the simplest possible

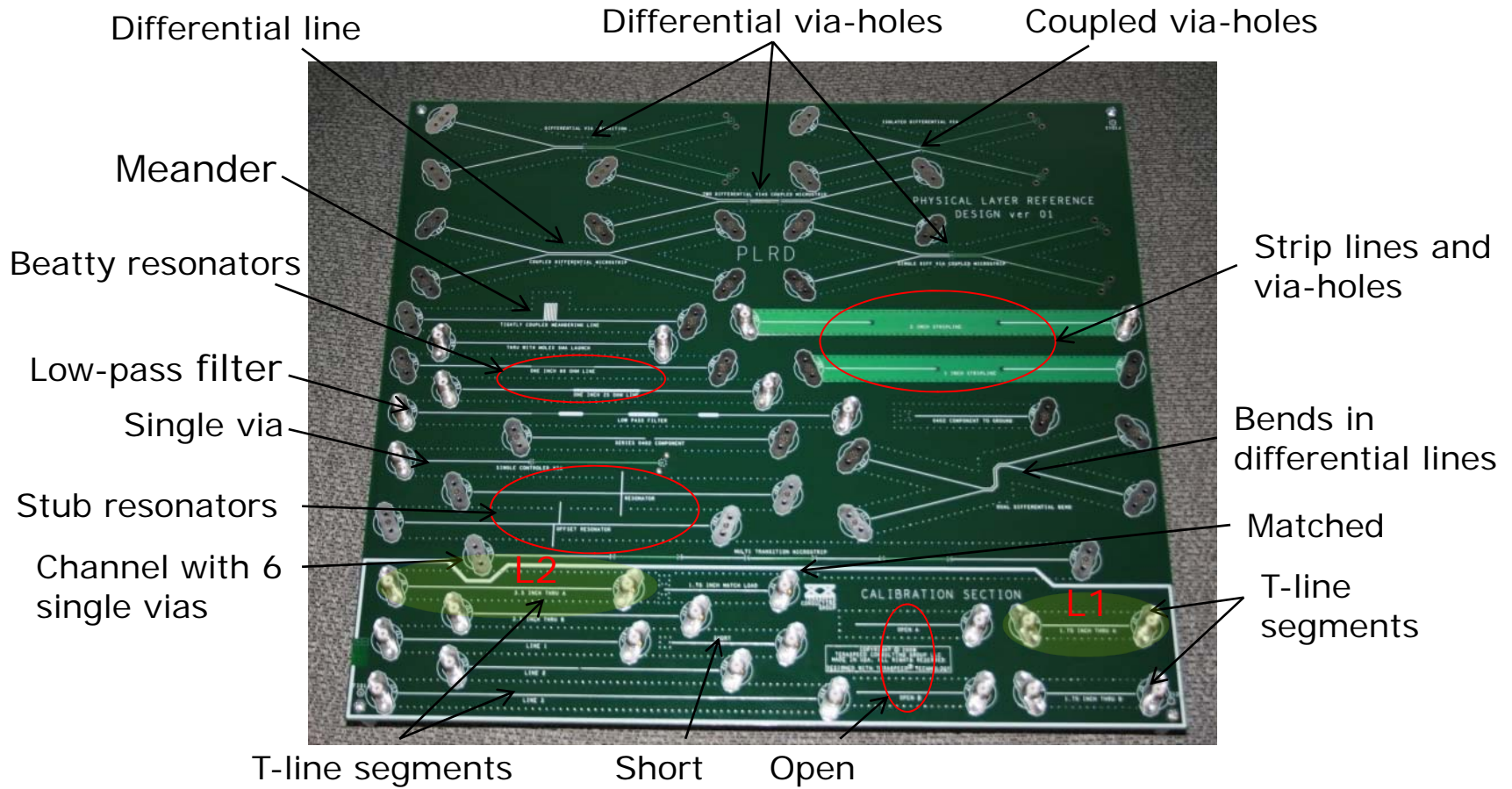
- ❑ Needs SOLT-calibrated measurements for 2 t-lines with any geometry of cross-section and transitions
 - No extraction of propagation constants (Γ) from measured data (difficult, error-prone)
 - No de-embedding of connectors and launches (difficult, error-prone)
- ❑ Needs the simplest numerical model
 - Requires computation of only propagation constants
 - No 3D electromagnetic models of the transitions
- ❑ Minimal number of smooth complex functions to match
 - One parameter for single and two parameters for differential
 - All reflection and modal transformation parameters are exactly zeros

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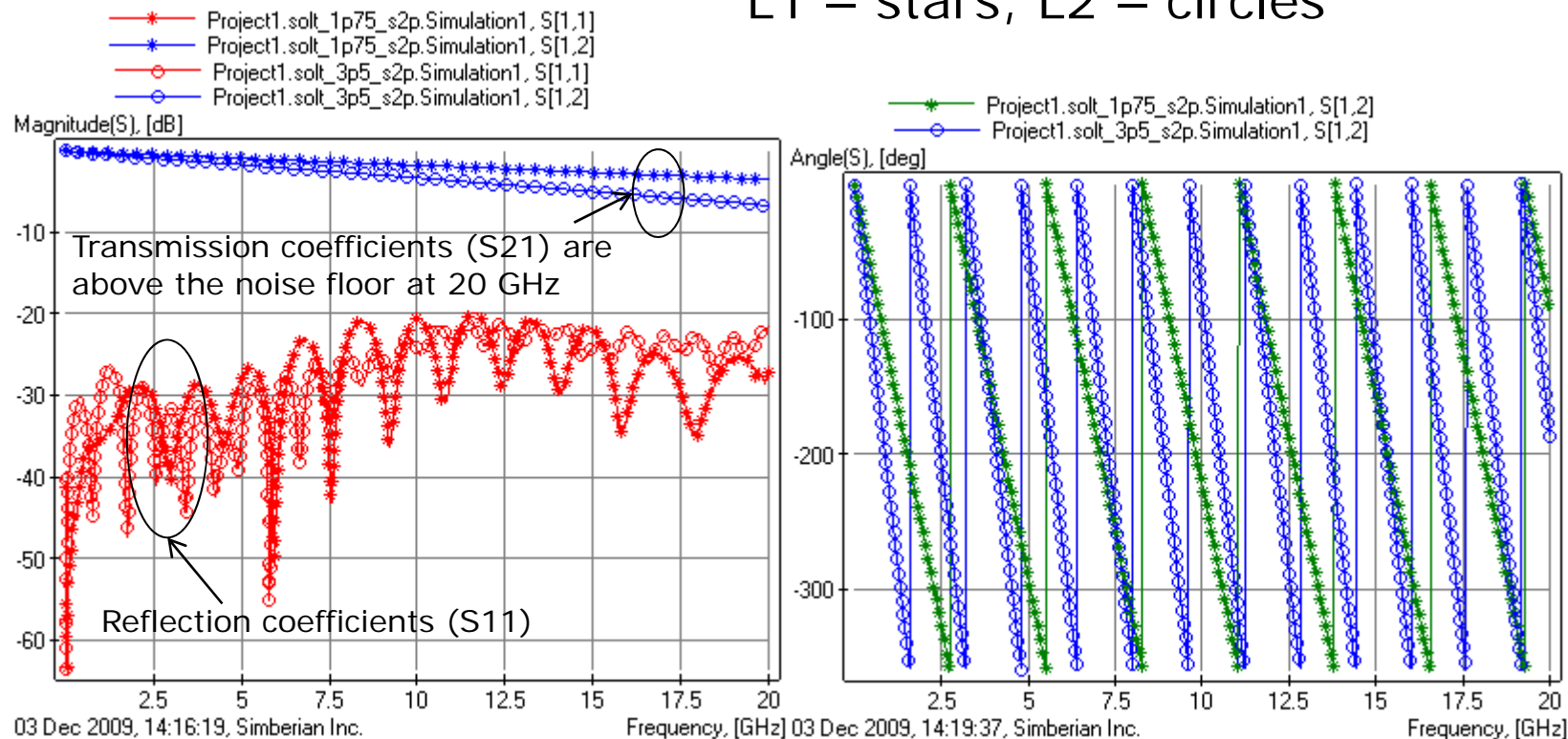
PLRD-1 Physical Layer Test Vehicle

- 30 test structures – all equipped with SMA connectors with optimized launch



S-parameters of line segments

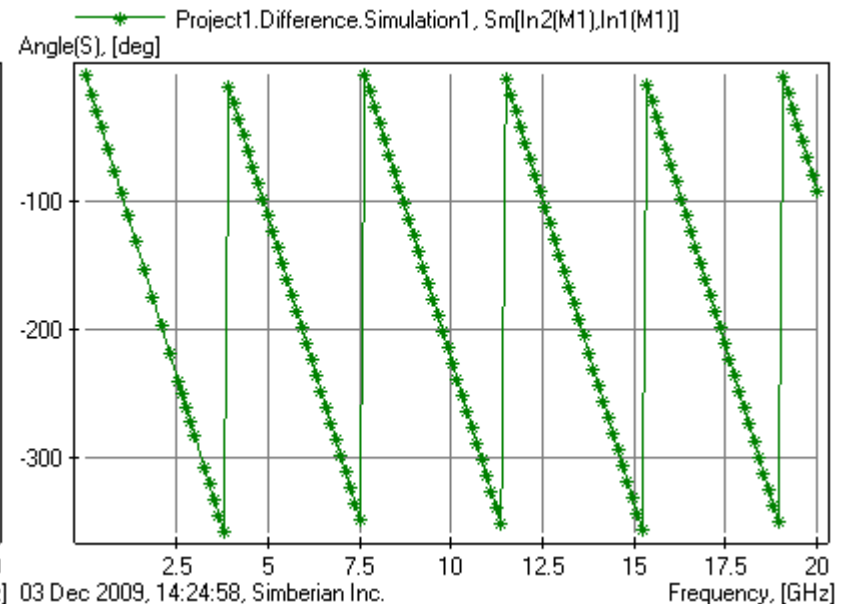
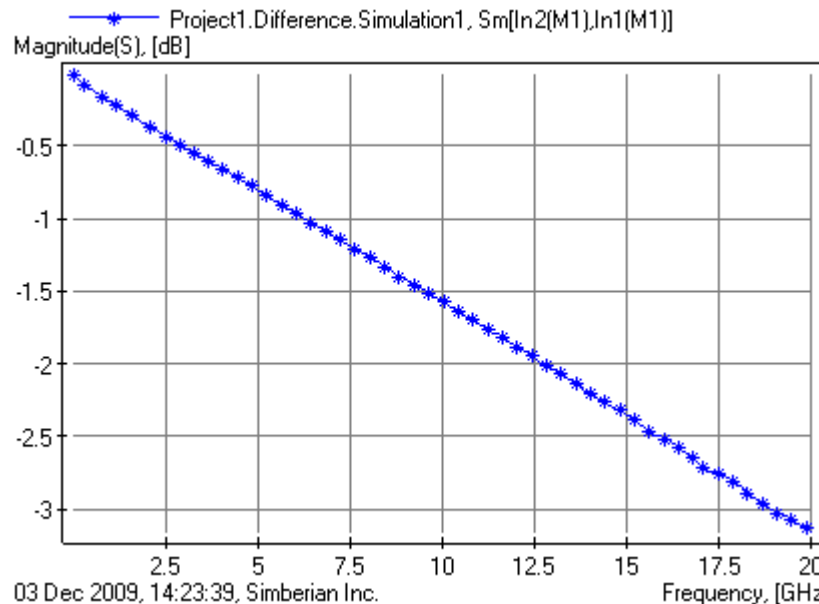
L1 – stars, L2 – circles



To use this data directly for the material parameters identification, we need to model transitions from/to the connectors or de-embed the launches

Generalized S-parameters of dL segment

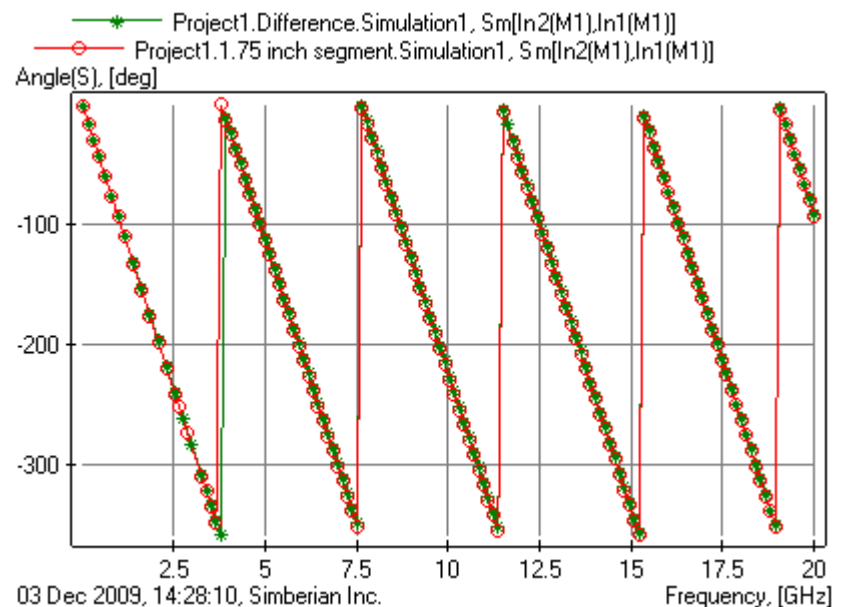
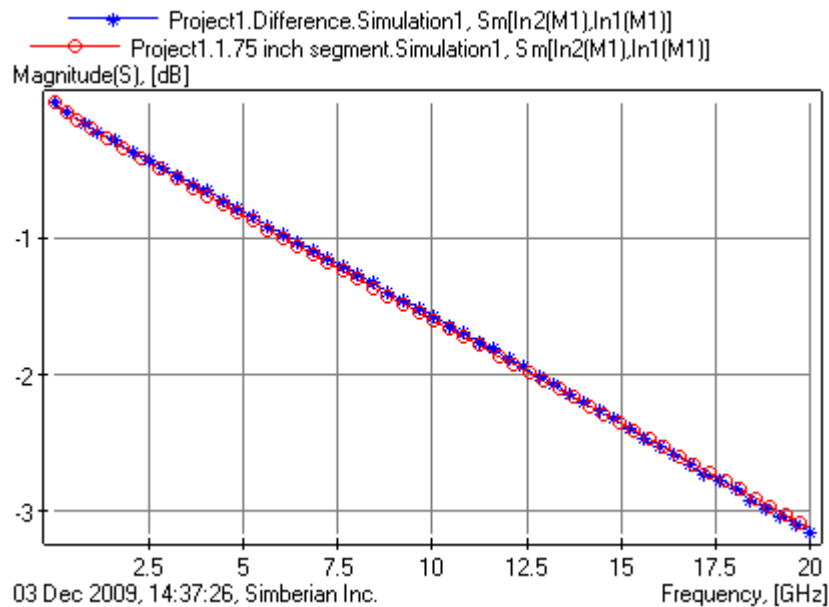
- Conversion to the modal generalized S-parameters gives us only the transmission coefficient ($S_{12}=S_{21}$) of the dominant micro-strip mode through the $dL=1.75$ inch segment



These data are suitable for the dielectric identification!

DK and LT identification by comparison

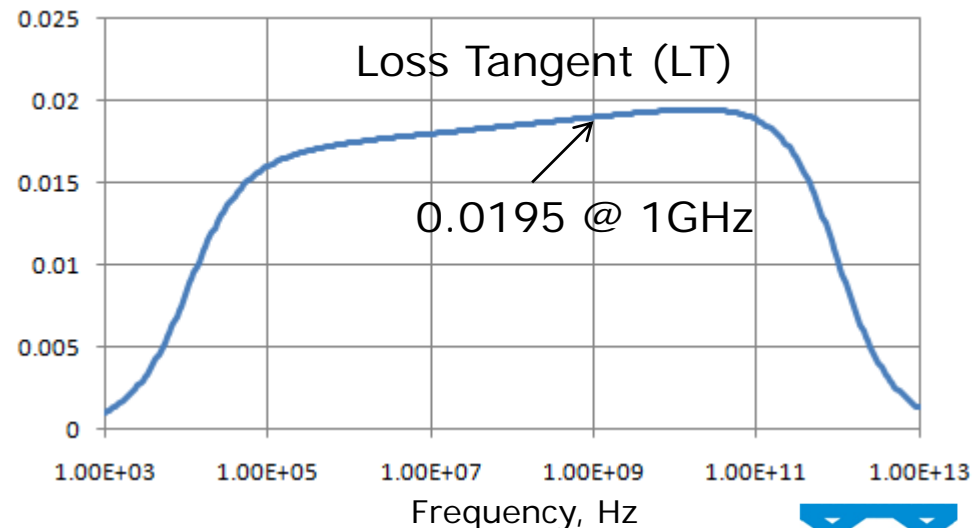
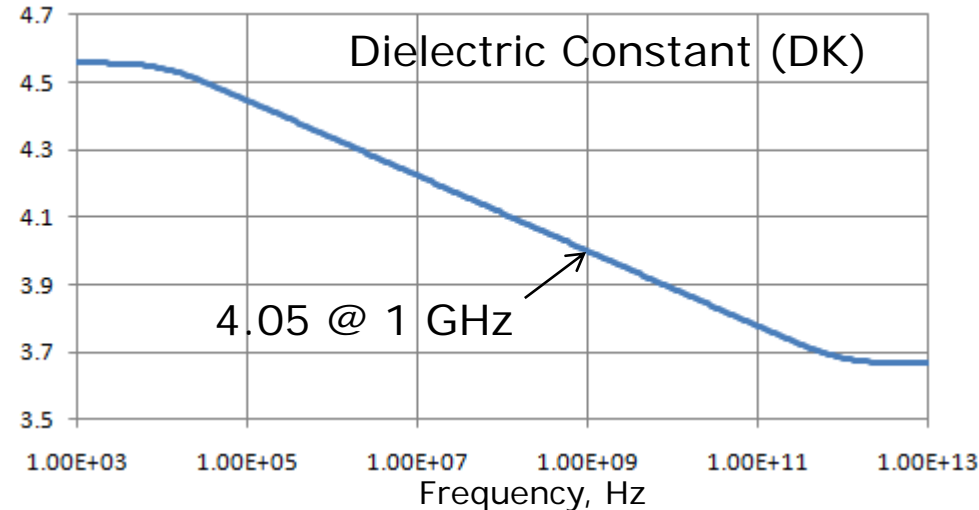
- Wide-band Debye dielectric model is used to compute S-parameters
- DK is adjusted to 4.05 @ 1 GHz to match the phase (right graph), and LT to 0.0195 @ 1GHz to match the attenuation (left graph)



Stars – measured and circles are simulated modal generalized S-parameters of 1.75 inch micro-strip line segment

Final dielectric model for PLRD-1 board

- Frequency-continuous wide-band Debye model
- DK is decreasing linearly on the log scale
- LT is almost constant
- Suitable for applications up to 100 Gb/s



Material A specifications and stackup for strip-line structures

- Core (measured with clamped strip-line, IPC TM-650, #2.5.5.5):
 - DK=3.48, LT=0.0037 @ 10 GHz, 0.0031 @ 2.5 GHz
 - DK=3.66 is recommended for analysis
- Prepreg:
 - DK=3.52, LT=0.004 @ 10 GHz

The screenshot shows a project stackup with the following layers and properties:

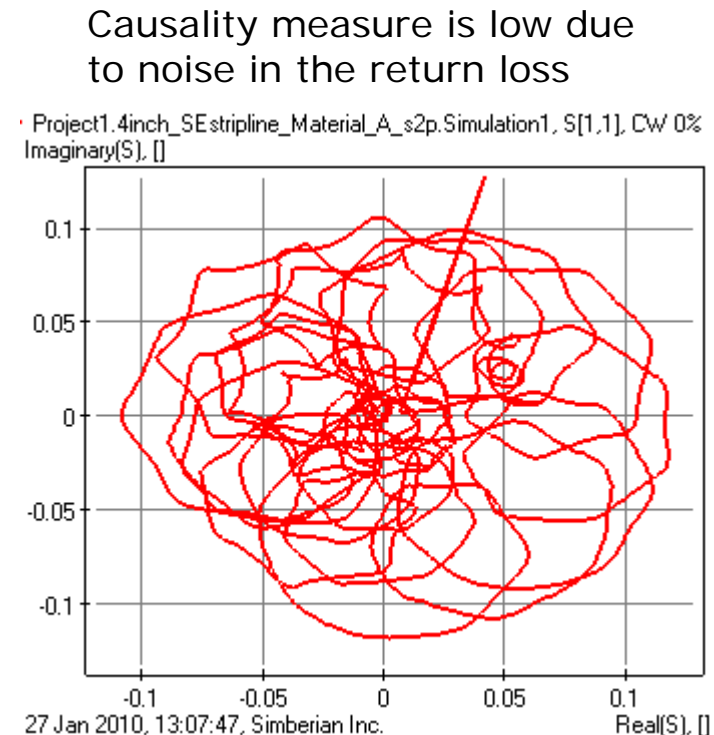
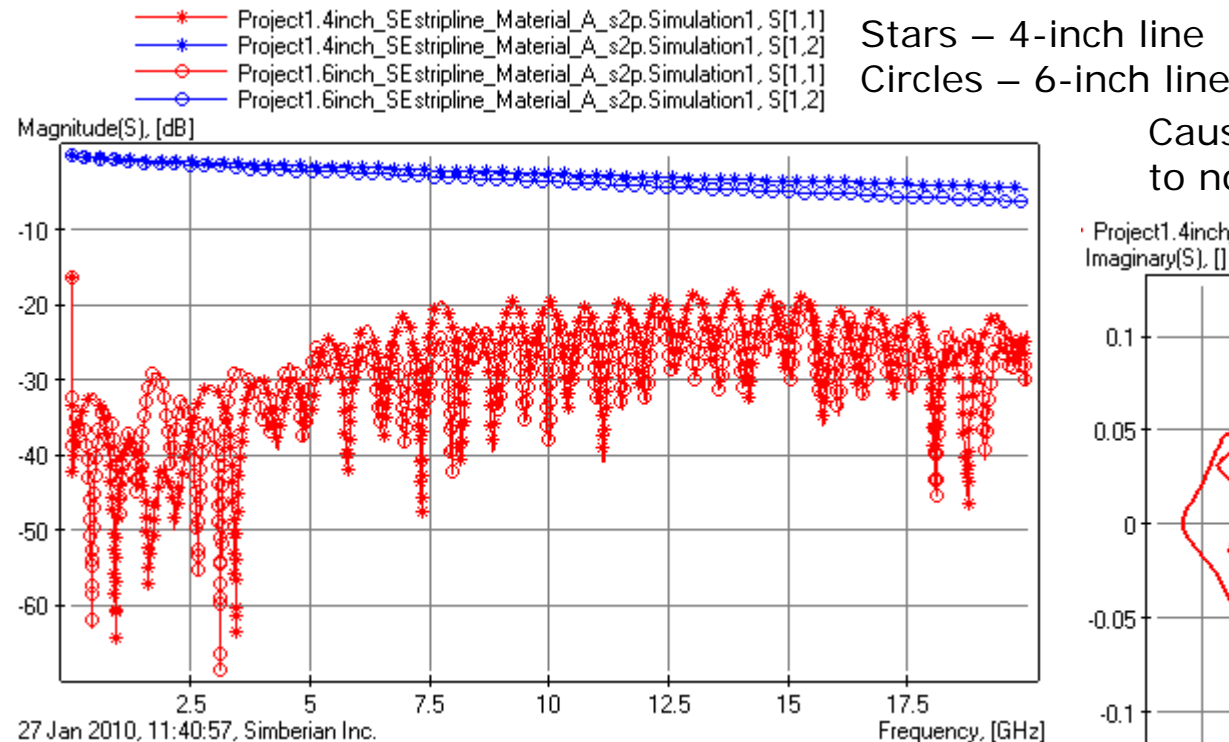
- Materials: T=20[°C], RF=2,...
- "Copper", RR=1, SR=0.25
- "Mat. A Core", Dk=3.66, LT=0.0037, PLM=WD
- "Mat. A Prepreg", Dk=3.52, LT=0.004, PLM=WD
- "Air"
- StackUp: LU=[mil], NL=3, T=18.5[mil]
- 1| Plane: "L1", Cond="Copper", T=2, Ins="Mat. A Prepreg", Rough
- 2| Medium: T=7.3, Ins="Mat. A Core"
- 3| Signal: "L2", T=0.6, Ins="Mat. A Prepreg", Rough
- 4| Medium: T=8, Ins="Mat. A Prepreg"
- 5| Plane: "L3", Cond="Copper", T=0.6, Ins="Mat. A Core", Rough

Annotations with arrows pointing to the stackup:

- Wideband Debye model is used first (points to the Core material)
- Roughness 0.3 um (points to the top copper plane)
- Bottom Roughness 0.5 um (points to the bottom copper plane)
- Top Roughness 0.7 um (points to the top copper plane)
- Roughness Factor 2 (points to the roughness settings)

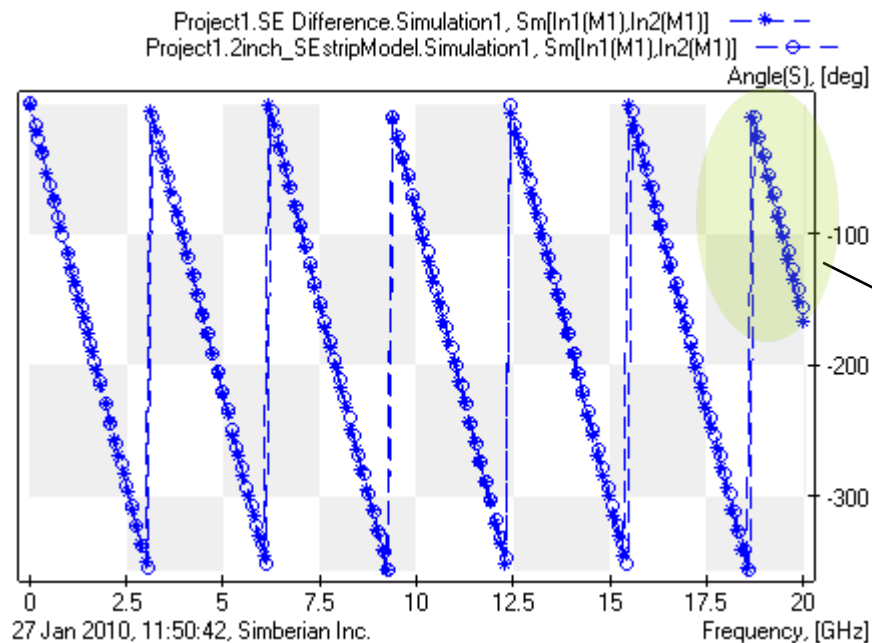
4 and 6 inch 50-Ohm strip-line segments with optimized launch

- 4-inch segment (stars): PQM=99.99%, RQM=96.27%, SQM=74%, CQM=0%
- 6-inch segment (circles): PQM=99.99%, RQM=96.8%, SQM=78%, CQM=0%

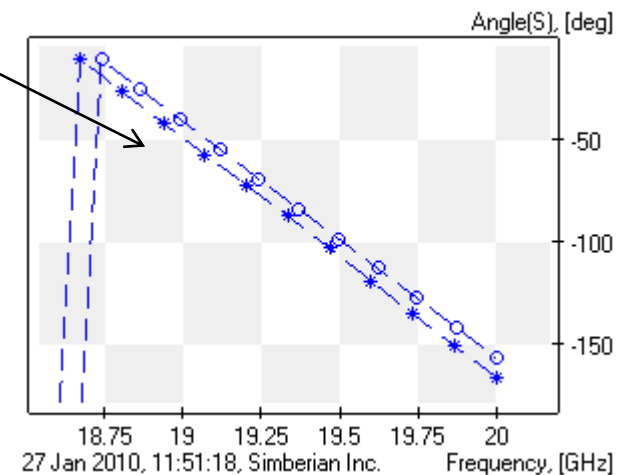


Generalized modal transmission phase

- Computed angle is consistently smaller over the frequency band



Stars - measured
Circles - computed

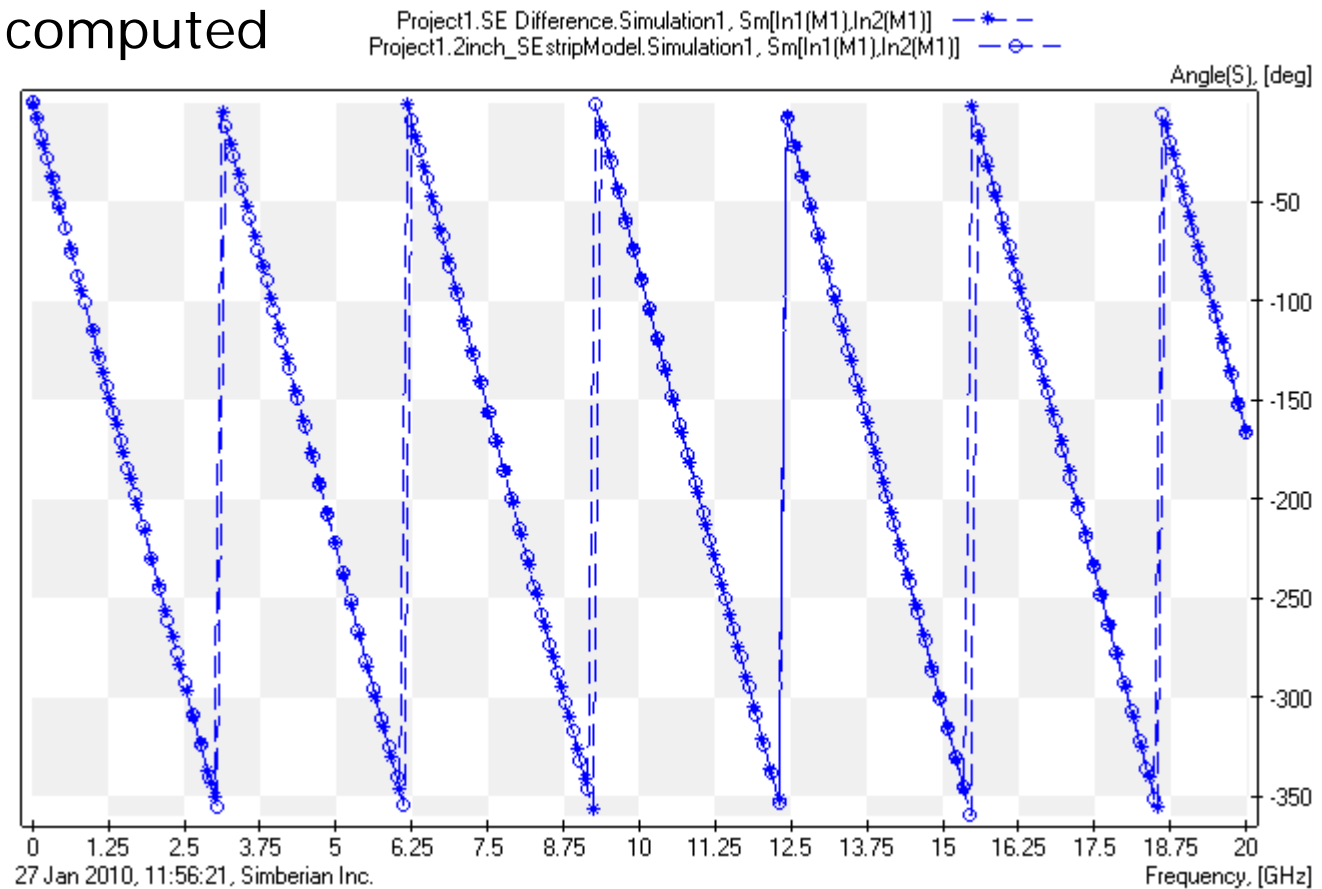


DK may be adjusted from 3.48 to 3.62 @ 10 GHz for core and prepreg for perfect match

Generalized modal transmission phase match with $DK=3.62$ @ 10 GHz

Stars - measured

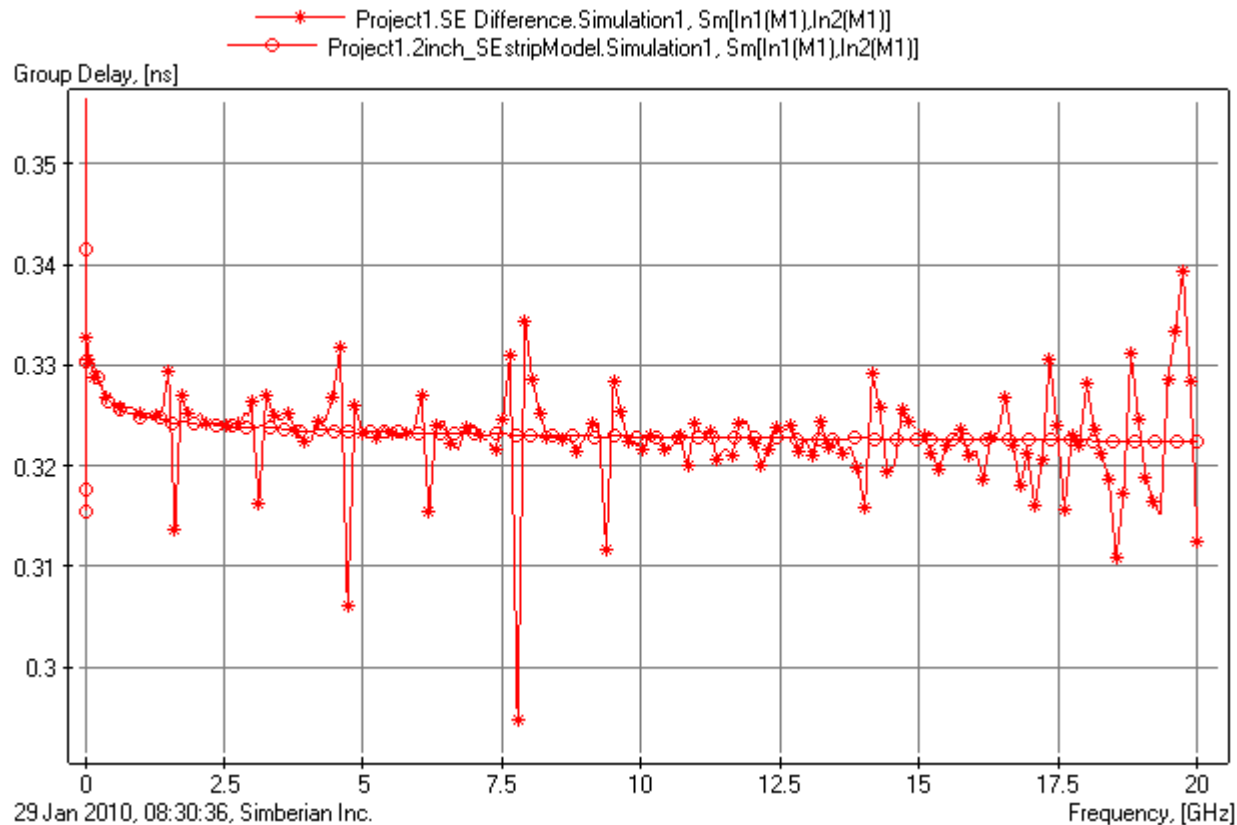
Circles – computed



Generalized modal group delay with DK=3.62 @ 10 GHz

Stars - measured

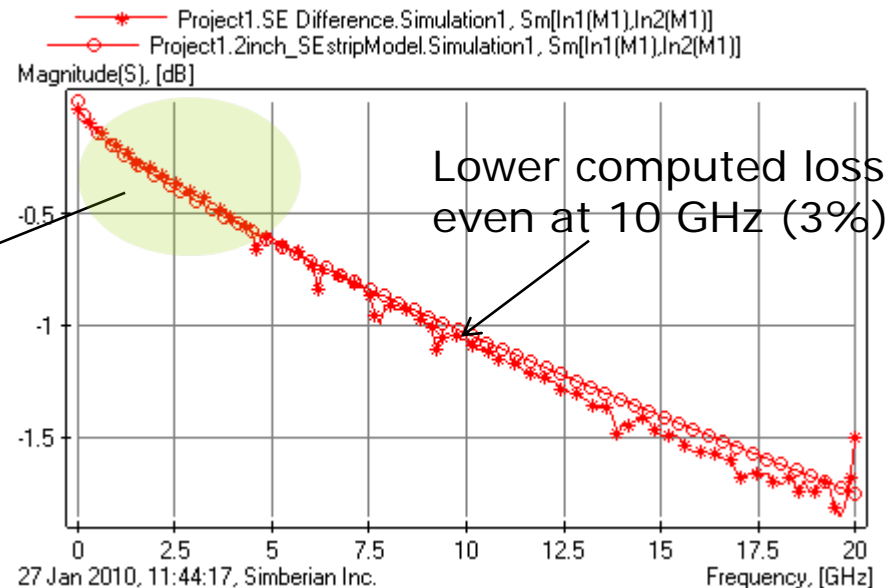
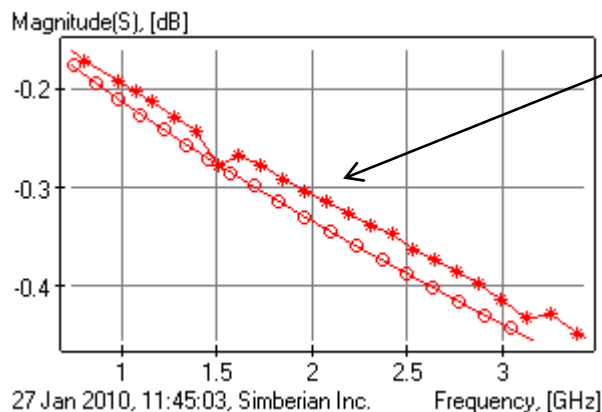
Circles – computed



Generalized modal attenuation parameter

- Computed attenuation is slightly larger than measured at lower frequencies and smaller at high frequencies

Stars - measured
Circles - computed



LT is smaller at lower frequencies and larger at higher
Roughness is probably too large too

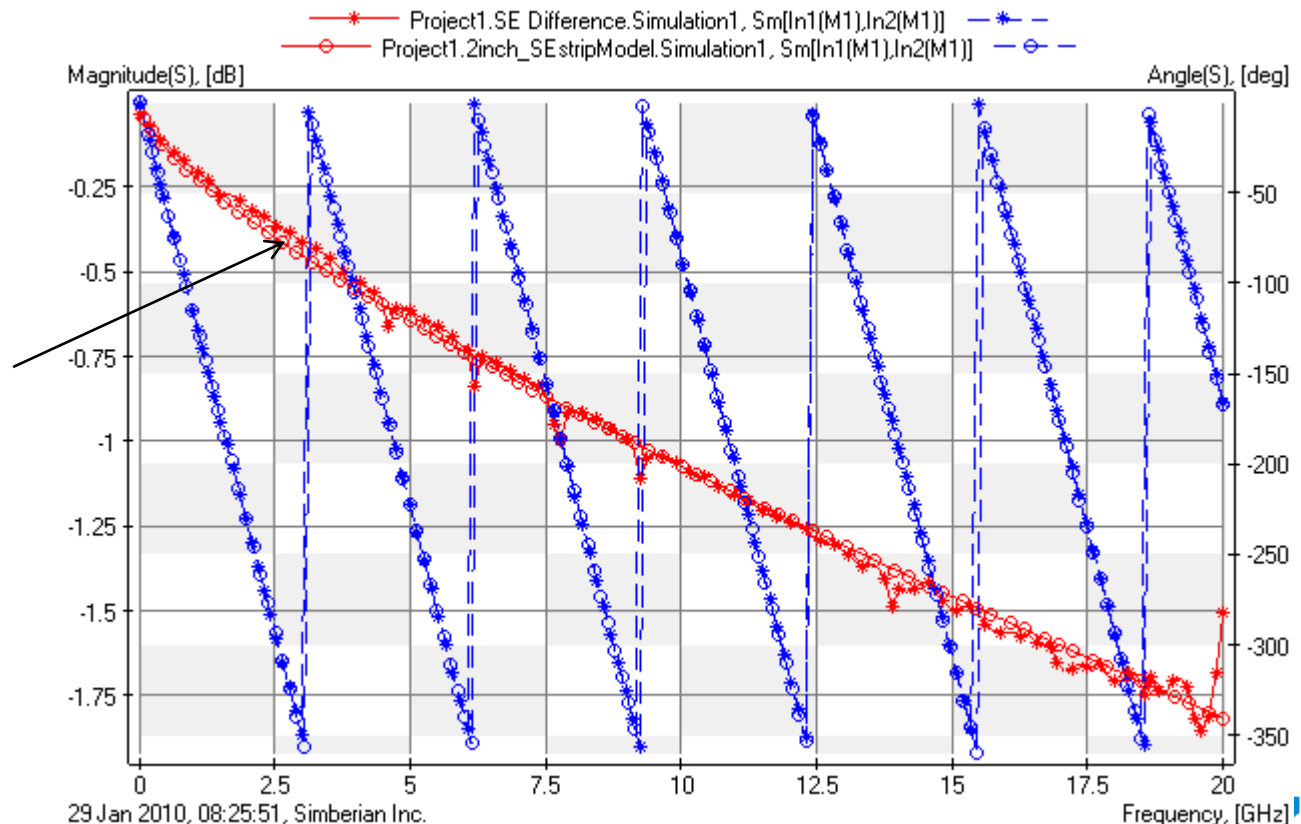
Possible WD model adjustment

- Use wideband Debye model with $DK=3.62$ and $LT=0.0038$ @ 10 GHz both for core and prepreg
- Adjust roughness to 0.5 μm ($RF=2$) for all conductor surfaces

Stars - measured
Circles - computed

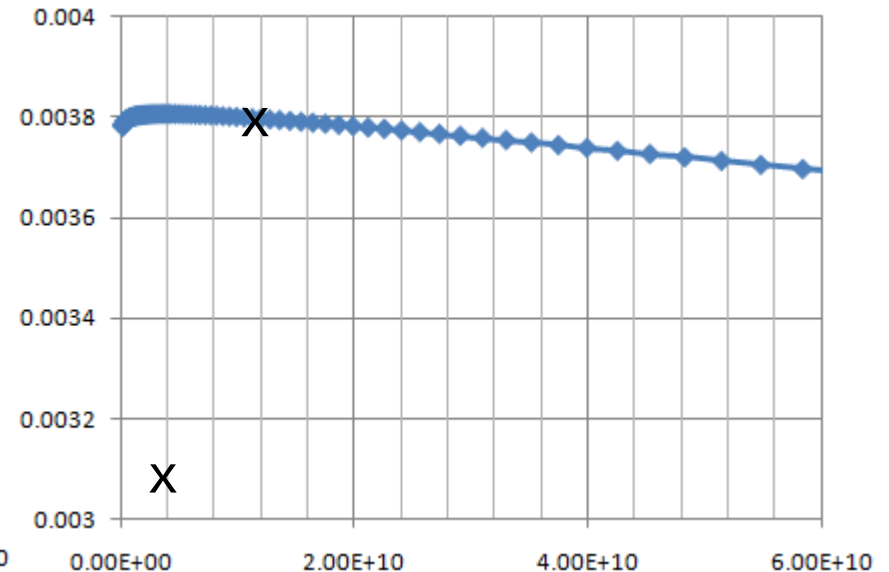
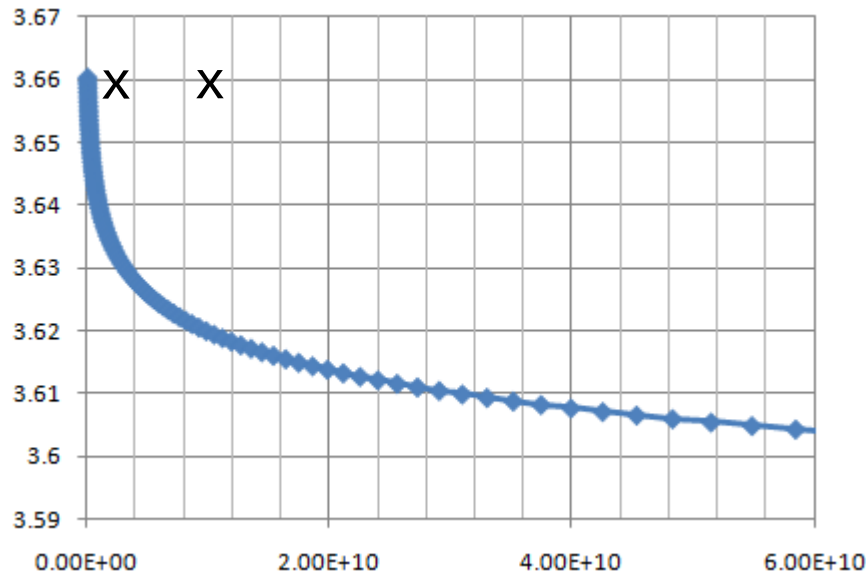
Minor deviation at lower frequencies

Phase and group delays are not changed



Final Wideband Debye (WD) model

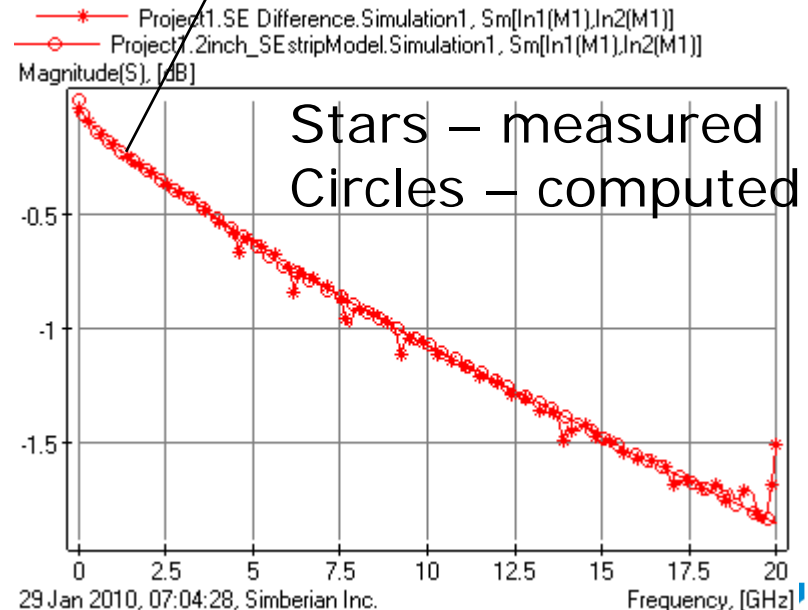
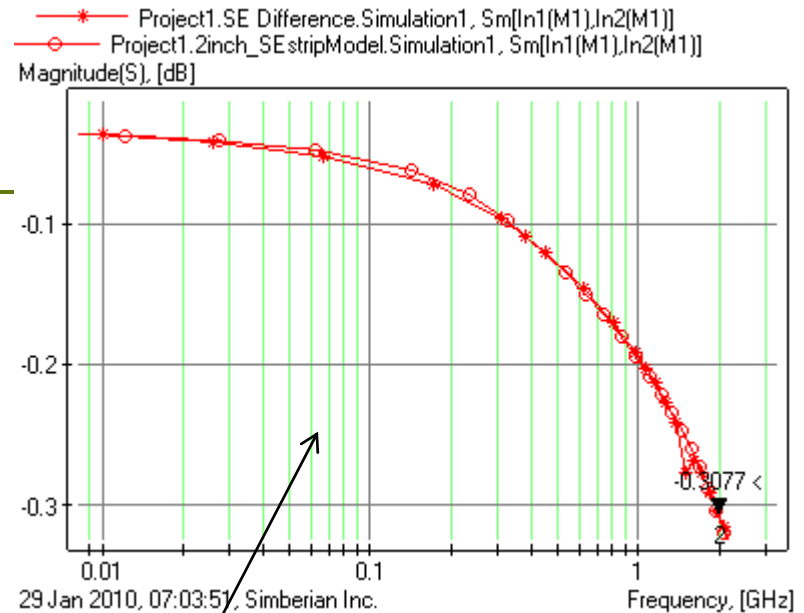
- Conductor model: $RR=1$; $SR=0.5 \text{ um}$; $RF=2$



x – values from dielectric specifications

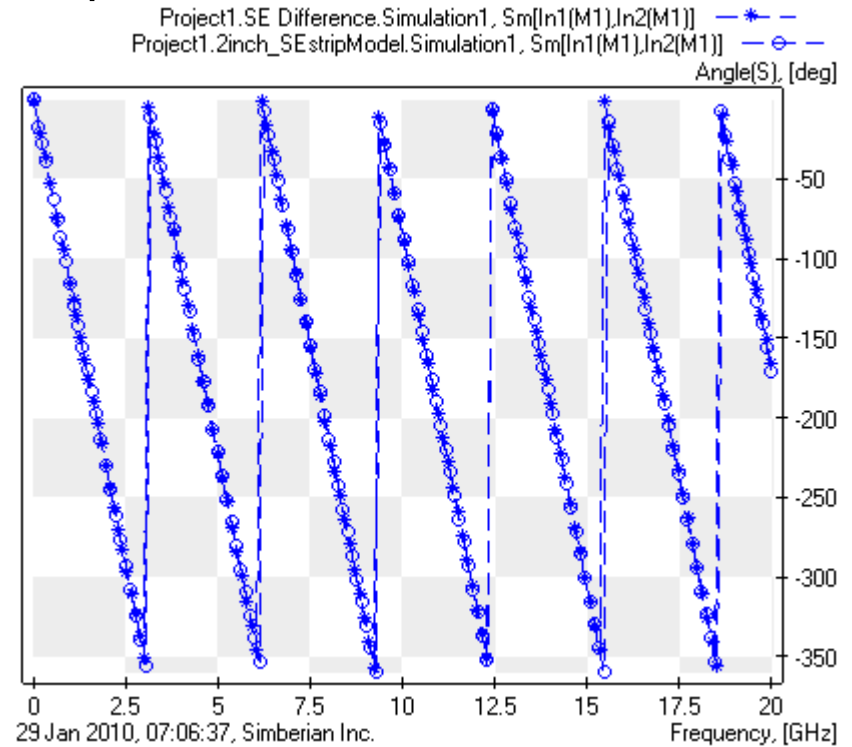
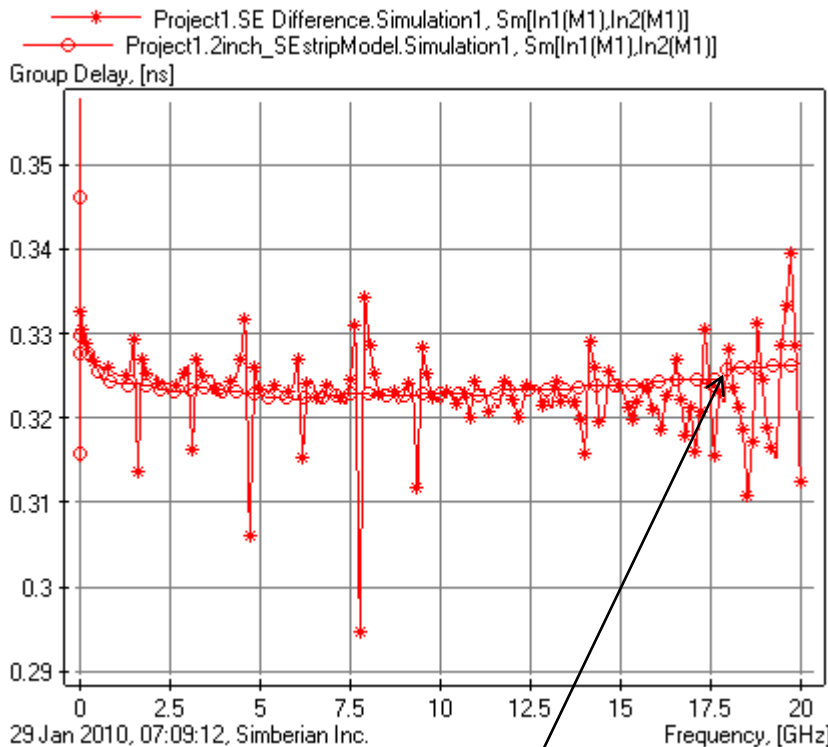
Model adjustments (2)

- To match DC to 1-2 GHz
 - Copper bulk resistivity relative to annealed copper is increased from 1 to 1.1
 - Roughness is adjusted to 0.25 μm , RF=2 for all surfaces
- Debye model with 16 poles constructed to match transmission from 1-2 GHz to 20 GHz



Phase and group delay for 16-pole Debye model

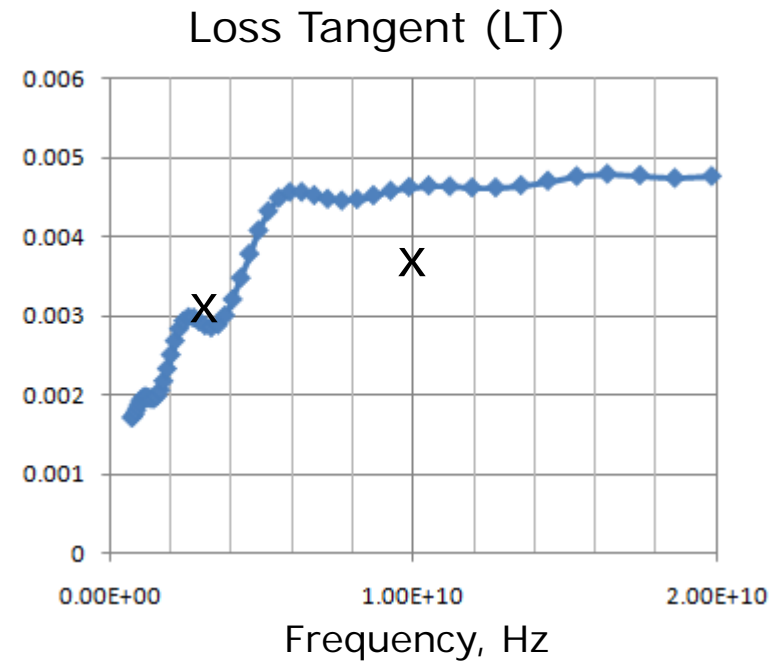
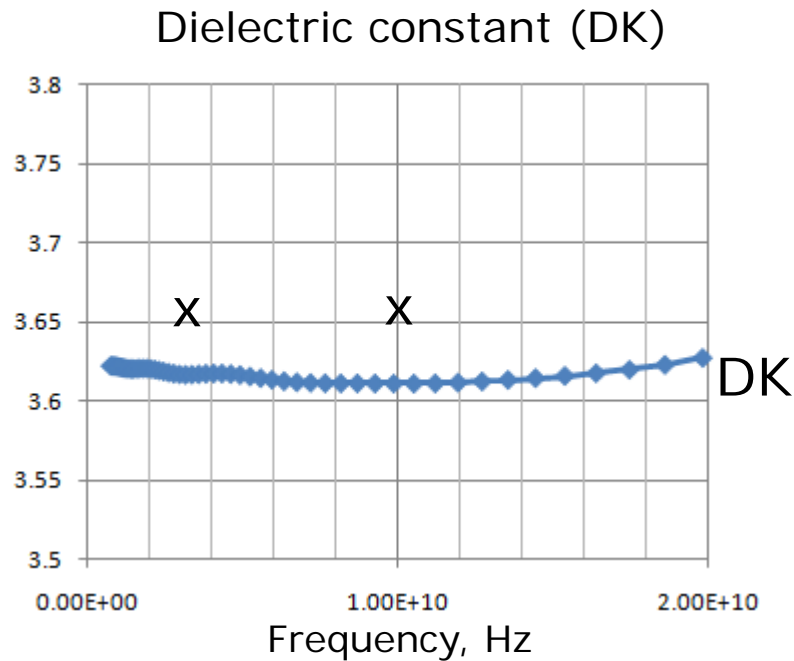
Stars – measured
 Circles – computed



Slight upward trend in the model

Final 16-pole Debye model

- Conductor model: $RR=1.1$; $SR=0.25$ μm ; $RF=2$



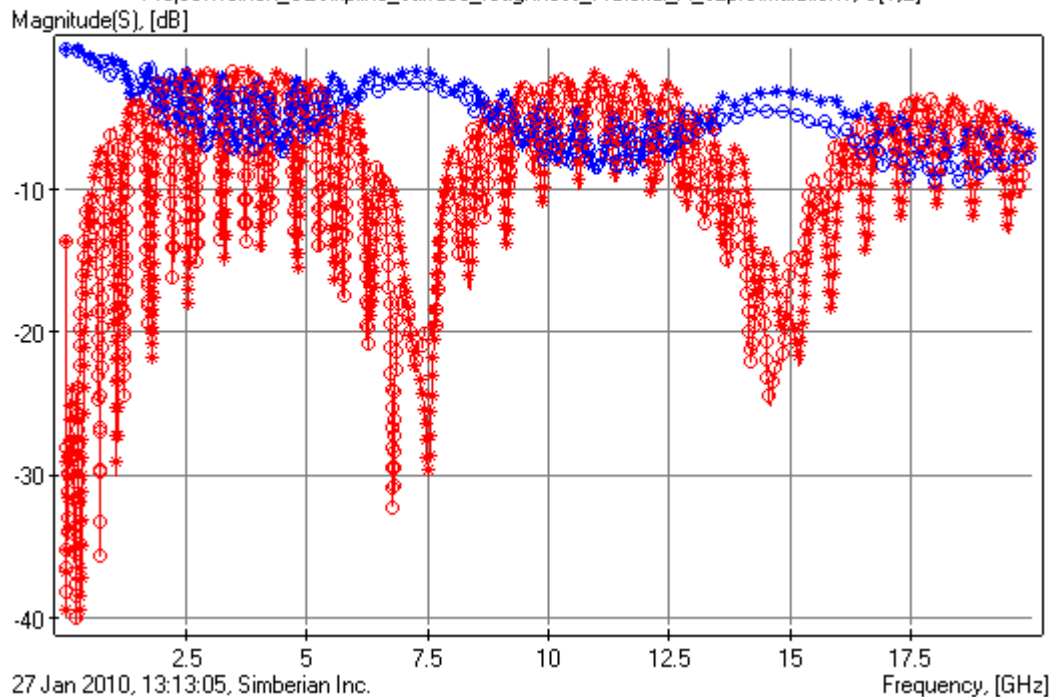
x – values from dielectric specifications

4 and 6 inch 25-Ohm strip-line segments (normalized to 25 Ohm)

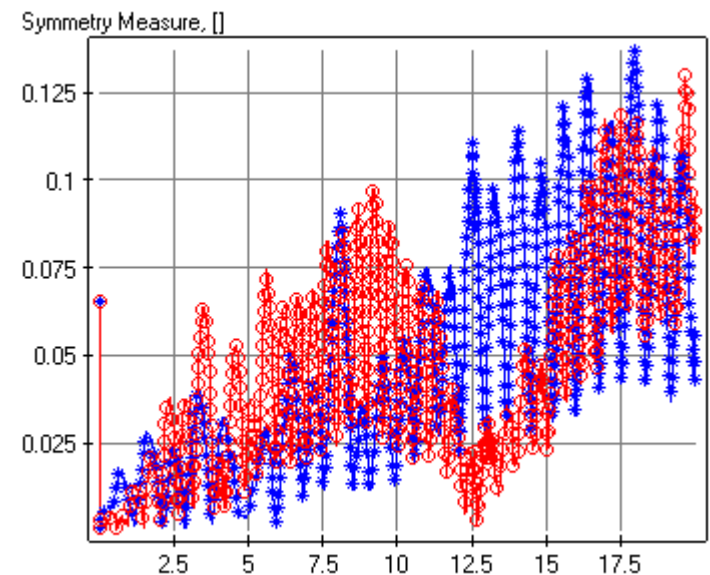
- 4-inch segment (stars): PQM=99.99%, RQM=96.3%, SQM=51%, CQM=77%
- 6-inch segment (circles): PQM=99.99%, RQM=96.6%, SQM=52%, CQM=88%

Project1.4inch_SEstripline_surface_roughness_Material_A_s2p.Simulation1, S[1,1]
Project1.4inch_SEstripline_surface_roughness_Material_A_s2p.Simulation1, S[1,2]
Project1.6inch_SEstripline_surface_roughness_Material_A_s2p.Simulation1, S[1,1]
Project1.6inch_SEstripline_surface_roughness_Material_A_s2p.Simulation1, S[1,2]

Stars – 4-inch line
Circles – 6-inch line

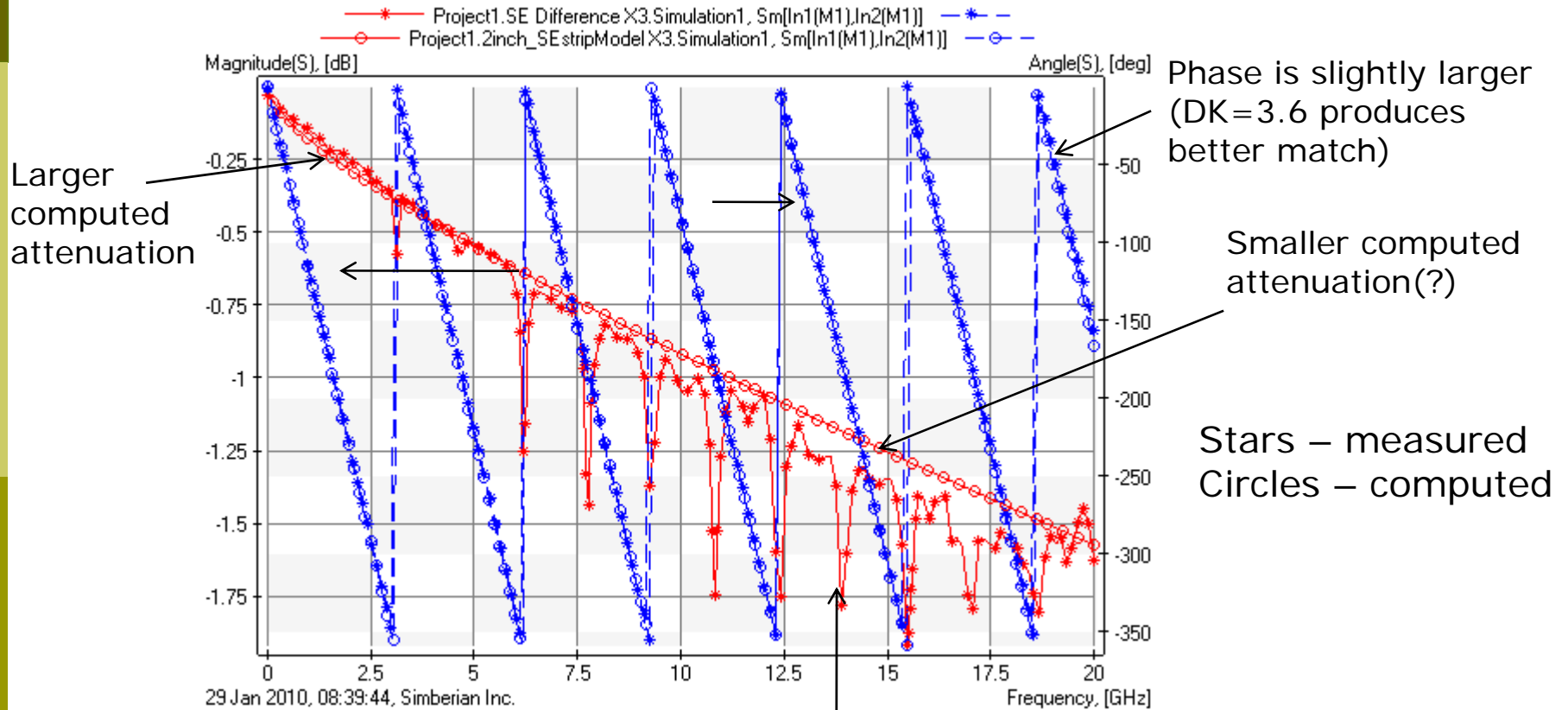


Symmetry quality measure is low
due to physical non-symmetry



Generalized modal transmission parameter for 25-Ohm strip-line

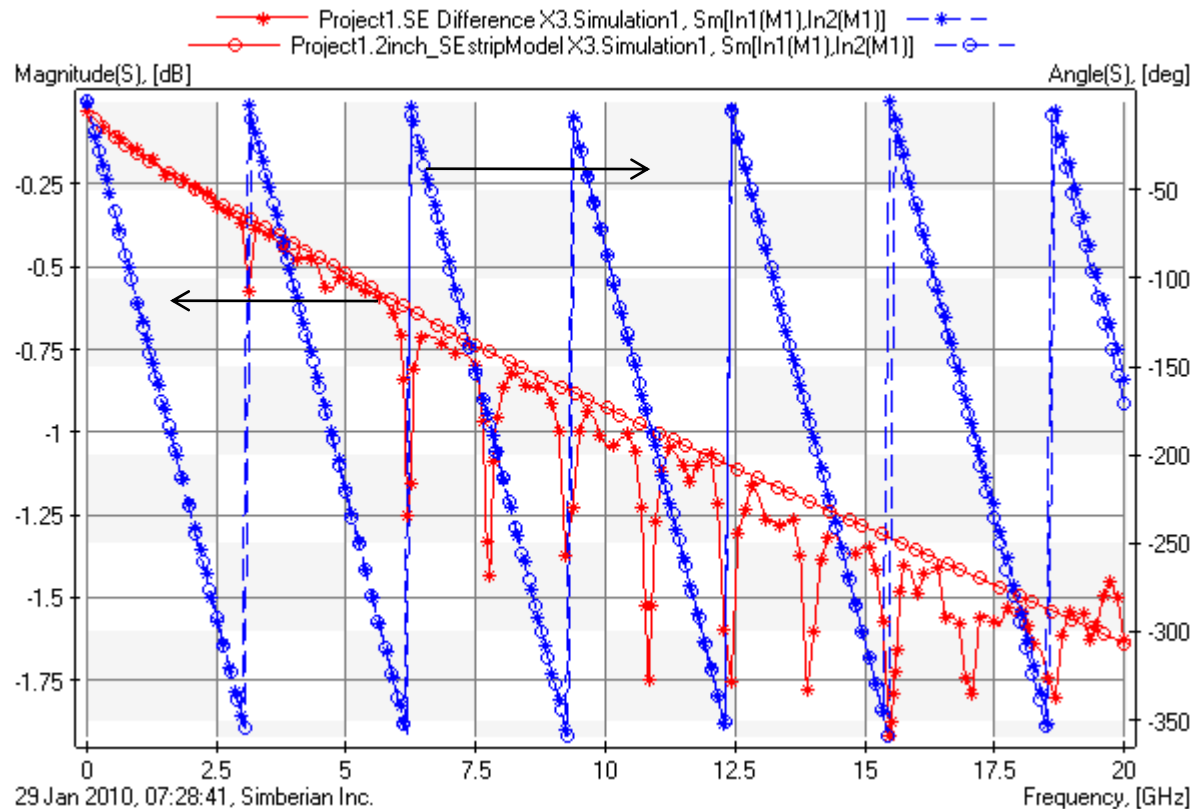
- Wideband Debye model: $DK=3.62$, $LT=0.0038$ @ 10 GHz
- Conductor: $RR=1.0$; $SR=0.5$ μm ; $RF=2$



Generalized modal transmission parameter for 25-Ohm strip-line

- 16-pole Debye model
- RR=1.1; SR=0.25 μm ; RF=2

Good correspondence at lower frequencies, not conclusive at high frequencies due to noise



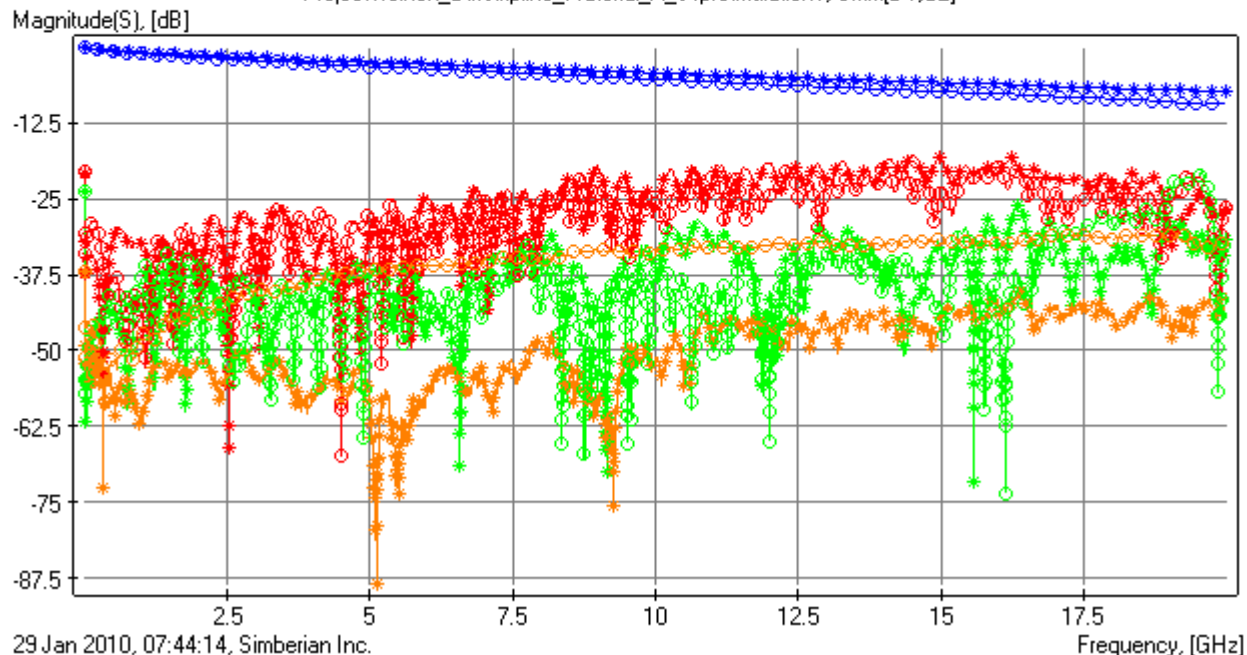
Stars – measured
Circles – computed

4 and 6 inch 100-Ohm differential strip-line segments

- 4-inch segment (stars): PQM=99.98%, RQM=99.1%, SQM=58%, CQM=9%
- 6-inch segment (circles): PQM=99.99%, RQM=99.3%, SQM=64%, CQM=17%

Project1.4inch_Diffstripline_Material_A_s4p.Simulation1.Smm[D1,D1]
Project1.4inch_Diffstripline_Material_A_s4p.Simulation1.Smm[D1,D2]
Project1.4inch_Diffstripline_Material_A_s4p.Simulation1.Smm[D1,C1]
Project1.4inch_Diffstripline_Material_A_s4p.Simulation1.Smm[D1,C2]
Project1.6inch_Diffstripline_Material_A_s4p.Simulation1.Smm[D1,D1]
Project1.6inch_Diffstripline_Material_A_s4p.Simulation1.Smm[D1,D2]
Project1.6inch_Diffstripline_Material_A_s4p.Simulation1.Smm[D1,C1]
Project1.6inch_Diffstripline_Material_A_s4p.Simulation1.Smm[D1,C2]

Stars – 4-inch line
Circles – 6-inch line

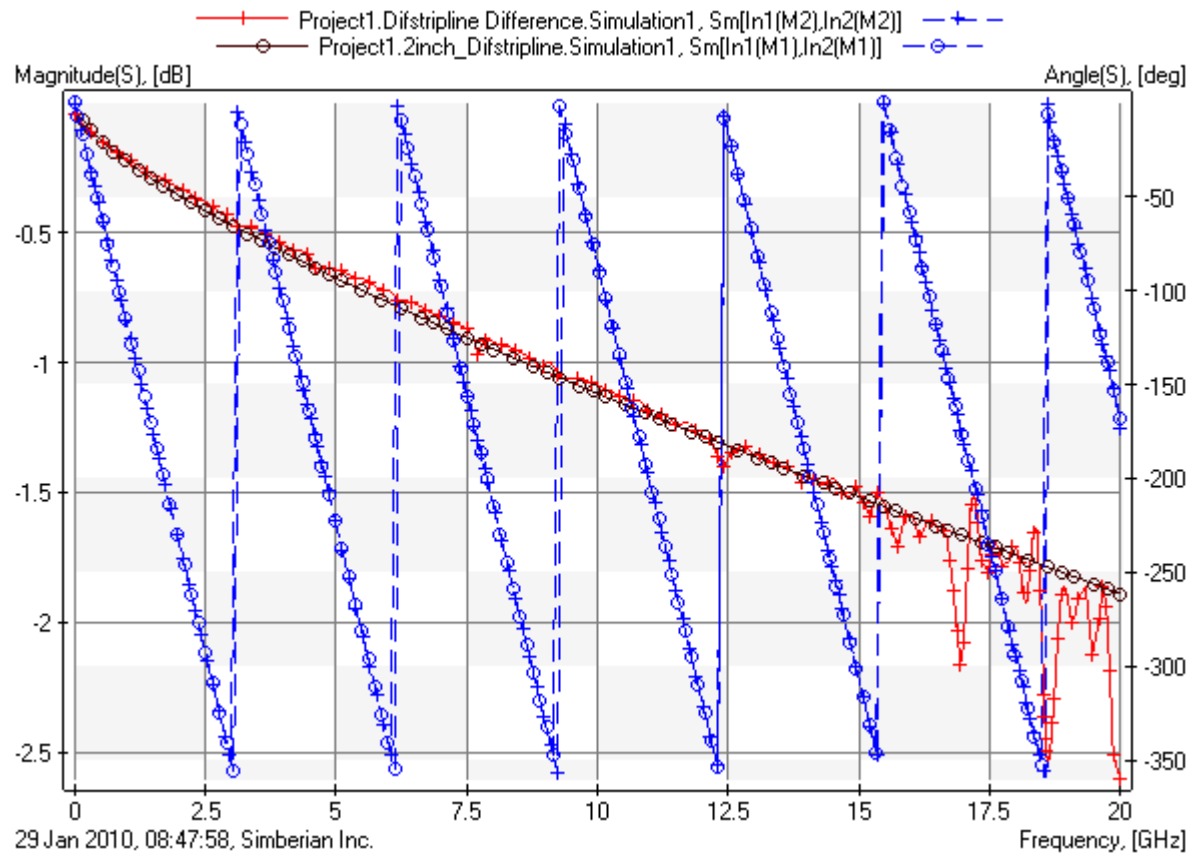


Mixed-Mode
S-parameters

29 Jan 2010, 07:44:14, Simberian Inc.

Generalized odd mode transmission parameter with WD model

- Wideband Debye model: $DK=3.62$, $LT=0.0038$ @ 10 GHz
- Conductor: $RR=1.0$; $SR=0.5$ μm ; $RF=2$

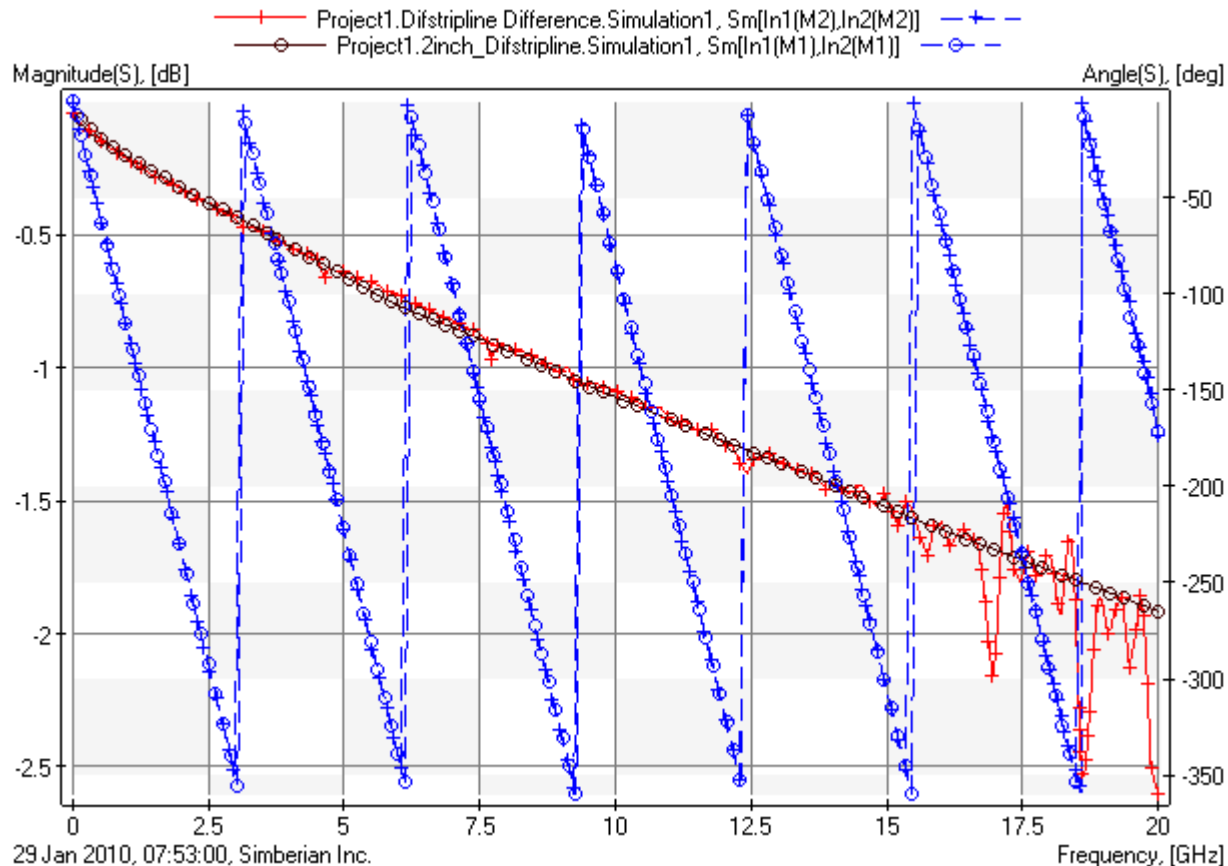


Pluses – measured;
Circles – computed;

Good
correspondence!

Generalized odd mode transmission parameter with 16-pole Debye model

- 16-pole Debye model
- RR=1.1; SR=0.25 μm ; RF=2

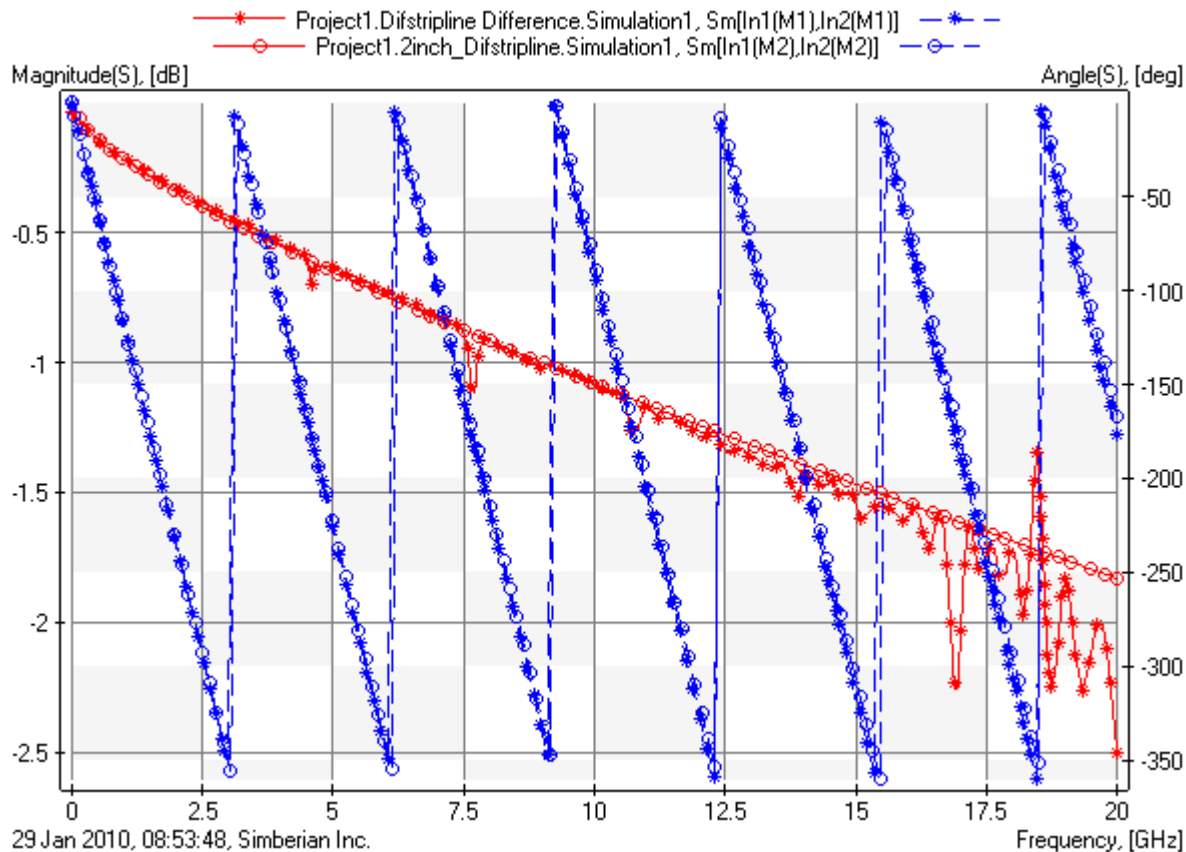


Pluses – measured;
Circles – computed;

Good
correspondence!

Generalized even mode transmission parameter with WD model

- Wideband Debye model: $DK=3.62$, $LT=0.0038$ @ 10 GHz
- Conductor: $RR=1.0$; $SR=0.5$ μm ; $RF=2$

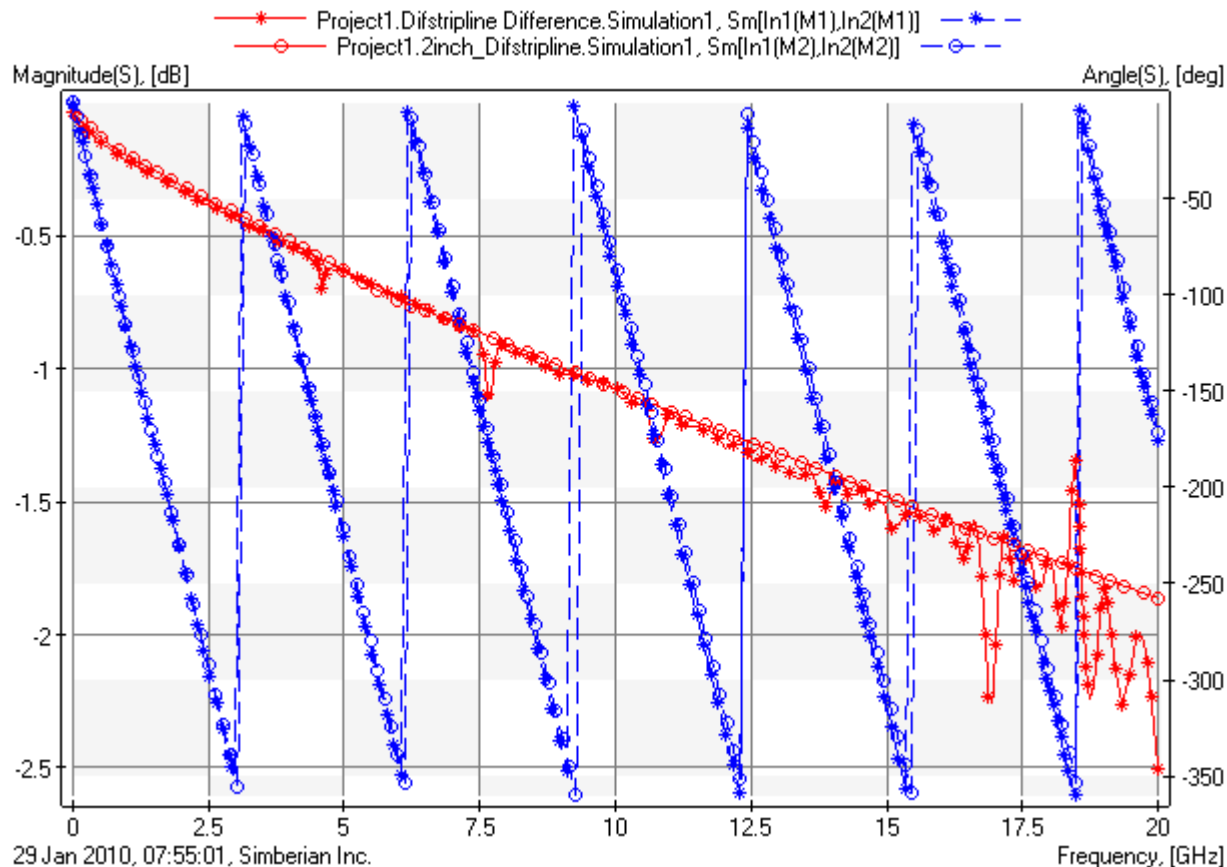


Stars – measured;
Circles – computed;

Some discrepancies
at high frequencies

Generalized even mode transmission parameter with 16-pole Debye model

- 16-pole Debye model
- RR=1.1; SR=0.25 μm ; RF=2

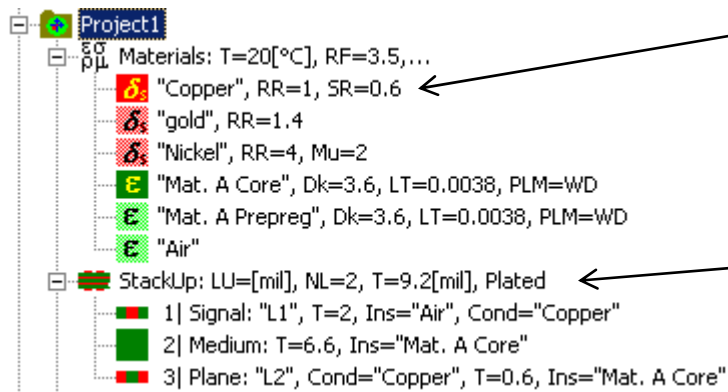


Stars – measured;
Circles – computed;

Some discrepancies
at high frequencies

Material A specifications and stackup for micro-line structures

- Core (measured with clamped strip-line, IPC TM-650, #2.5.5.5):
 - DK=3.48, LT=0.0037 @ 10 GHz, 0.0031 @ 2.5 GHz
 - DK=3.66 is recommended for analysis
- Prepreg:
 - DK=3.52, LT=0.004 @ 10 GHz

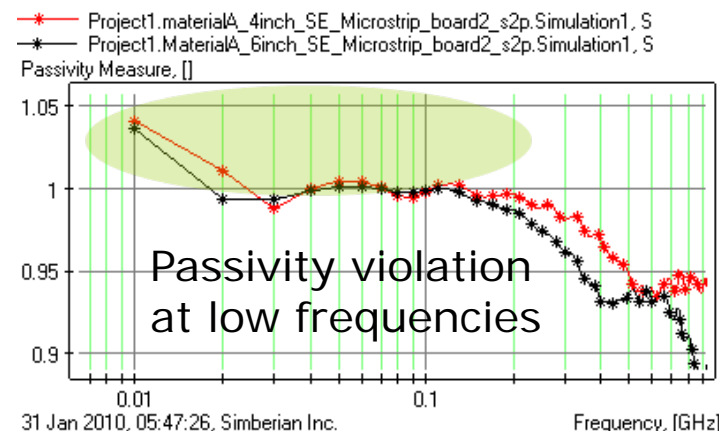
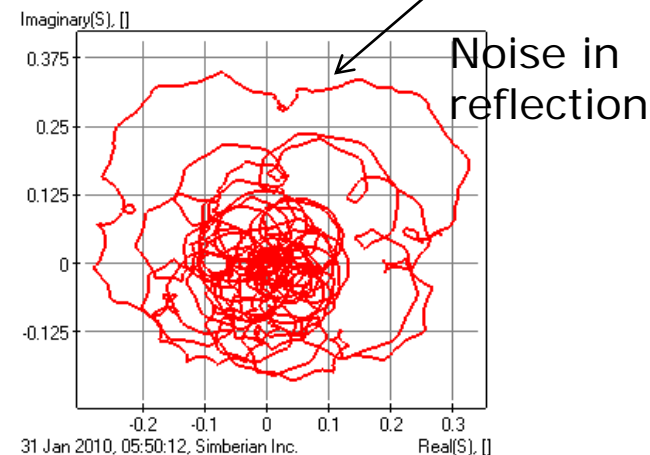
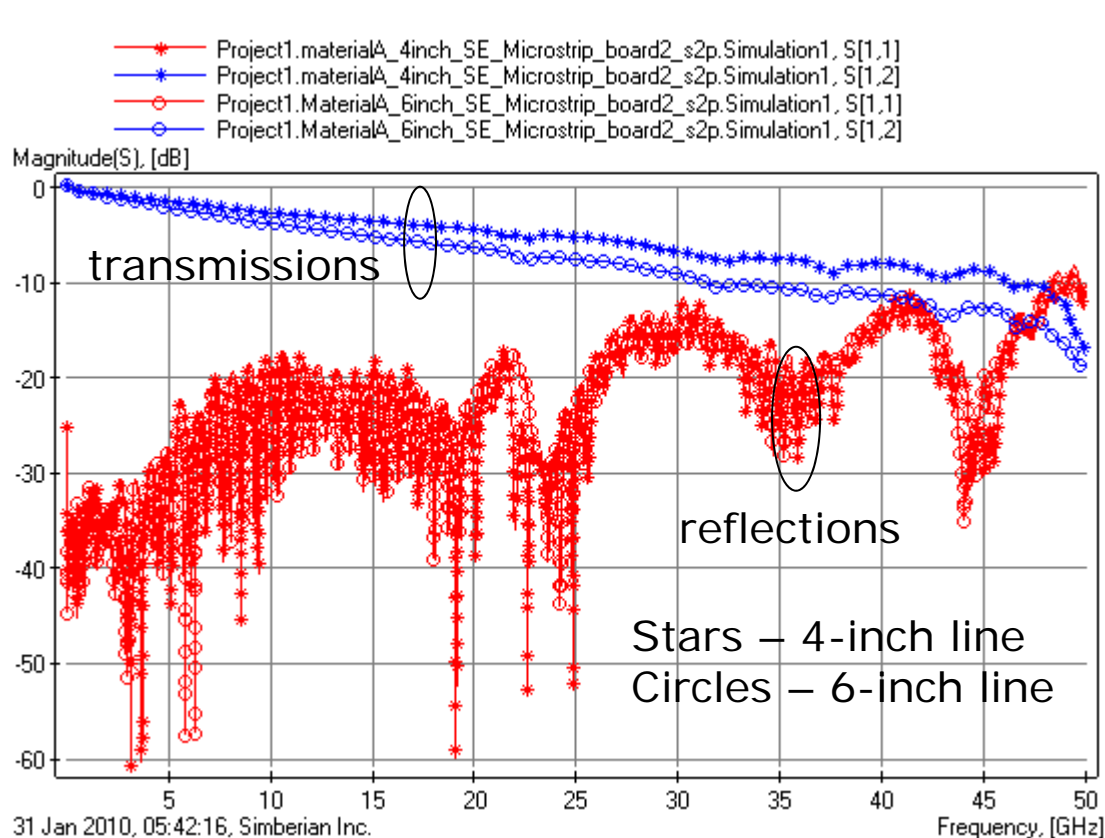


Common SR=0.6, RF=3.5

Top layer is plated with 2.5 um of Nickel and 0.05 um of Gold

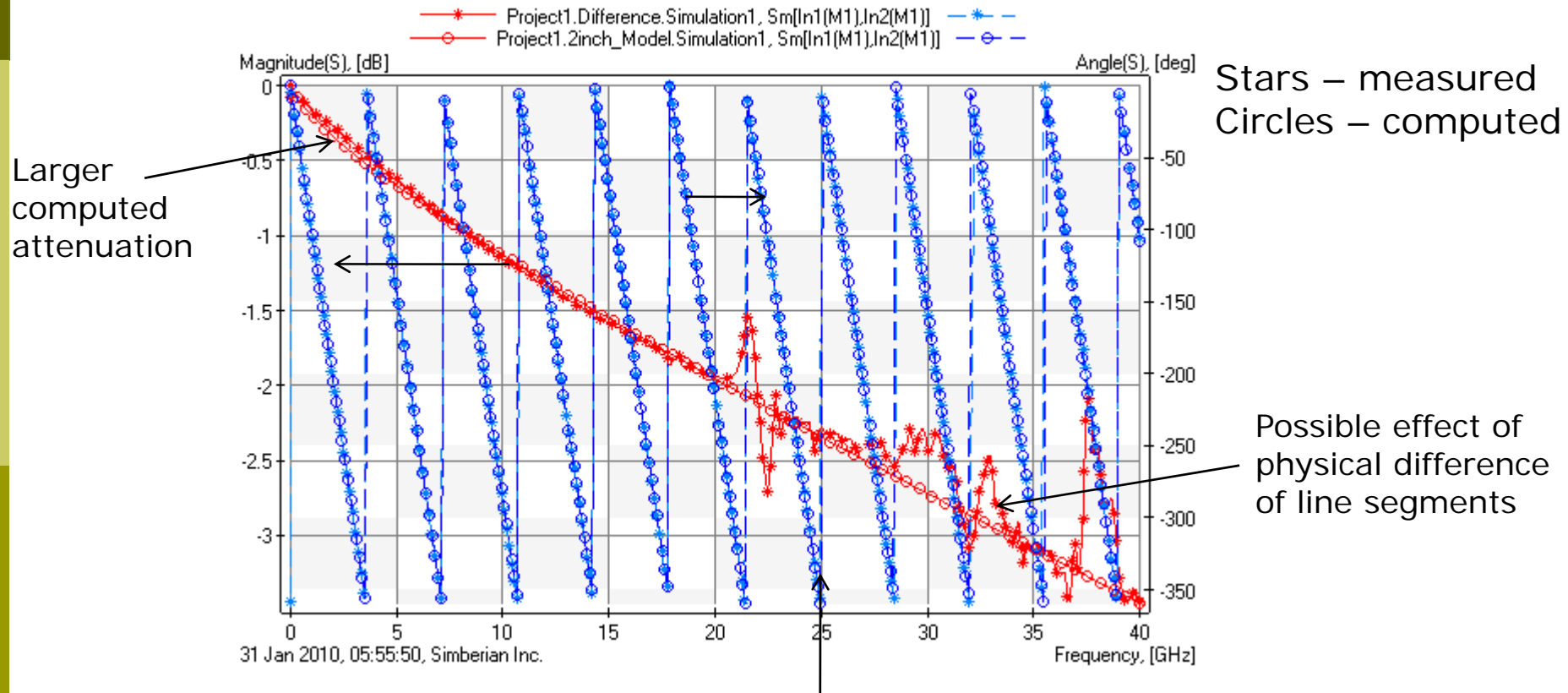
4 and 6 inch 50-Ohm micro-strip line segments

- 4-inch segment (stars): PQM=99.98%, RQM=99.0%, SQM=41%, CQM=0%
- 6-inch segment (circles): PQM=99.99%, RQM=99.3%, SQM=45%, CQM=0%



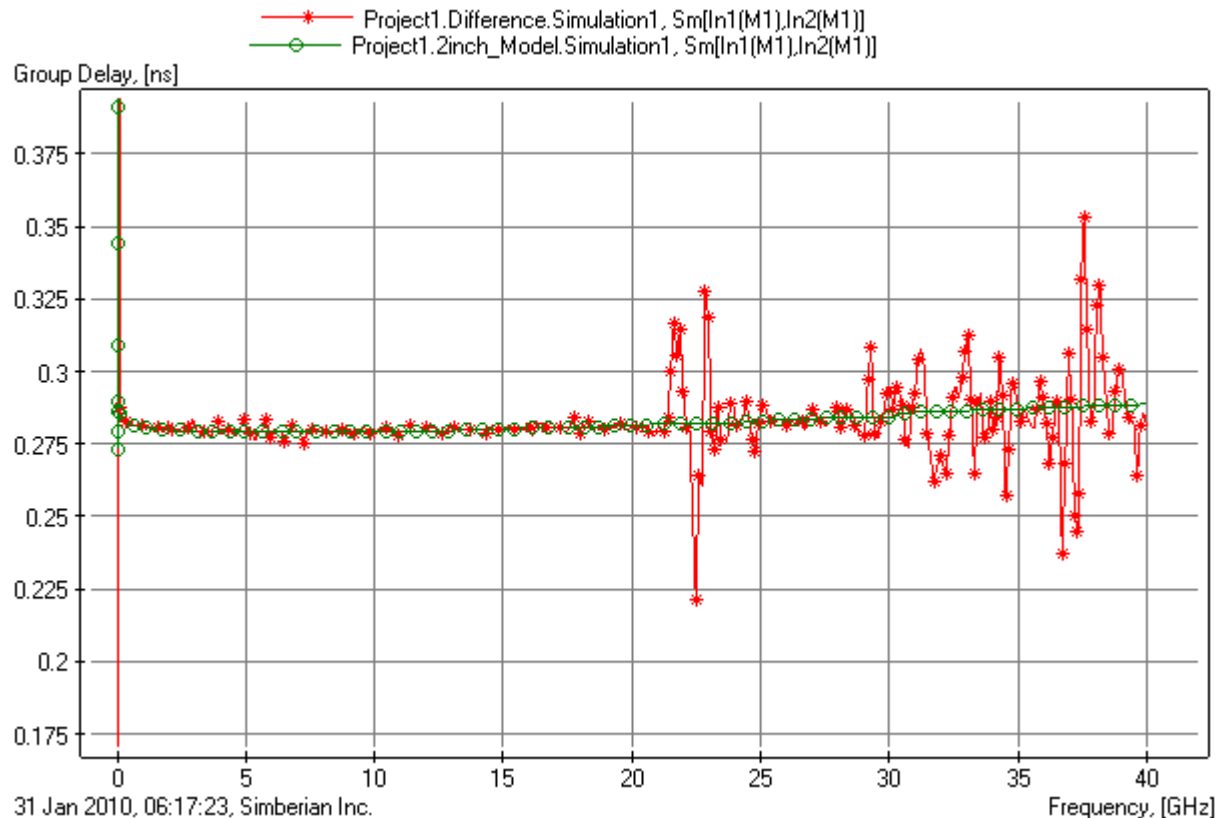
Generalized modal transmission parameter for 50-Ohm micro-strip line

- Wideband Debye model: $DK=3.6$, $LT=0.0038$ @ 10 GHz
- Conductor: plated copper; $SR=0.6$ μm ; $RF=3.5$



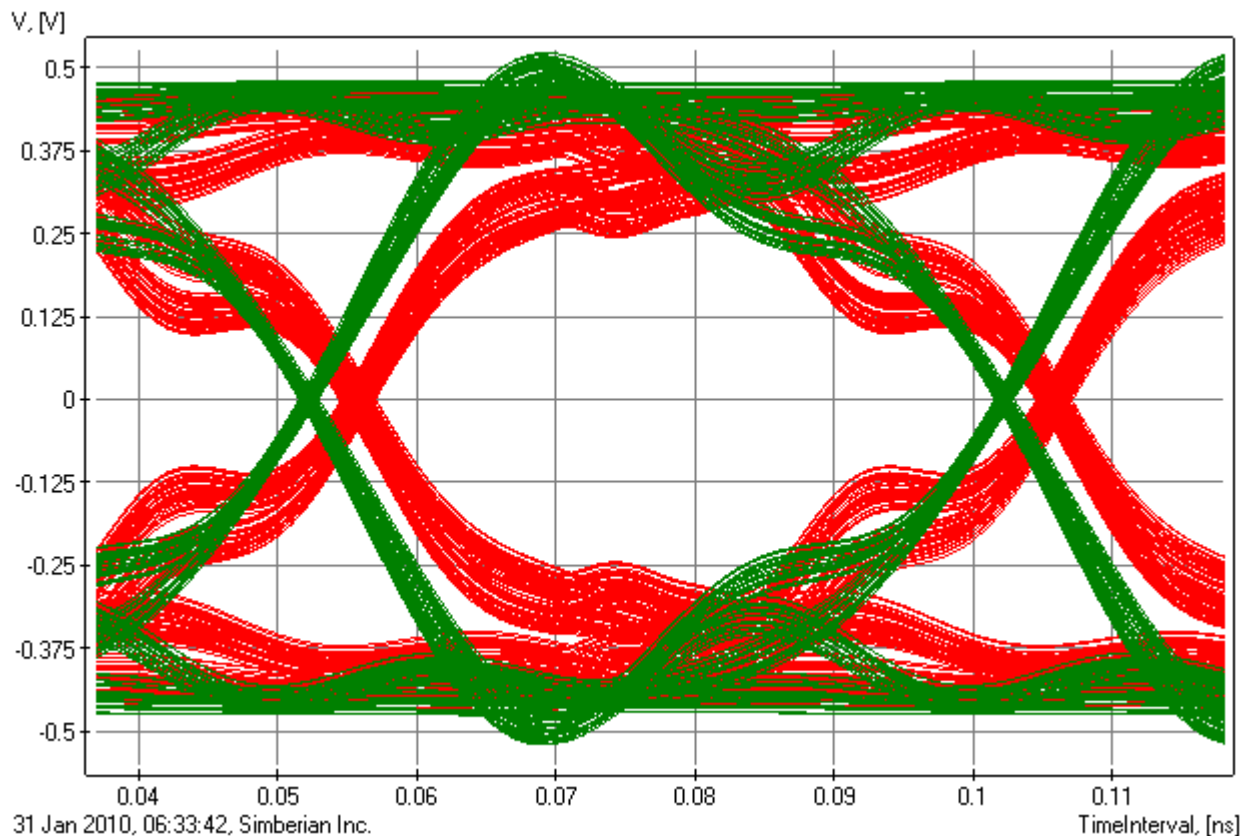
Group delay for 50-Ohm micro-strip line

- Wideband Debye model: $DK=3.6$, $LT=0.0038$ @ 10 GHz
- Conductor: plated copper; $SR=0.6$ μm ; $RF=3.5$



Importance of the roughness model

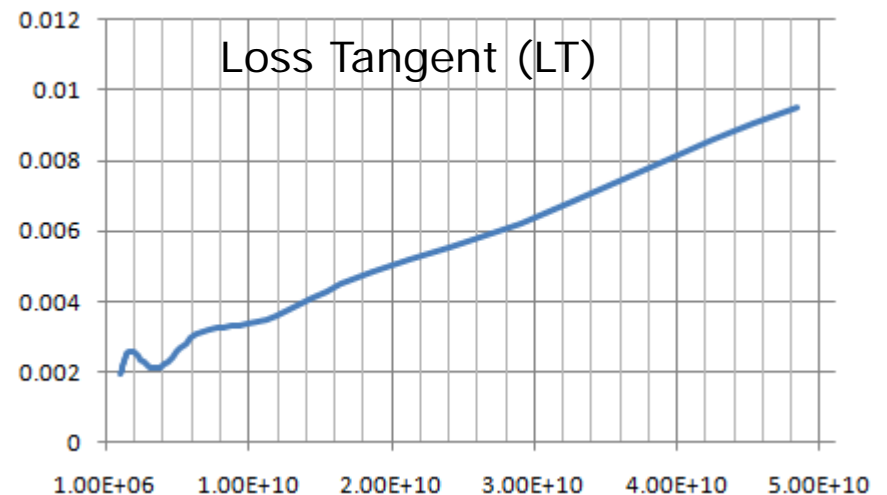
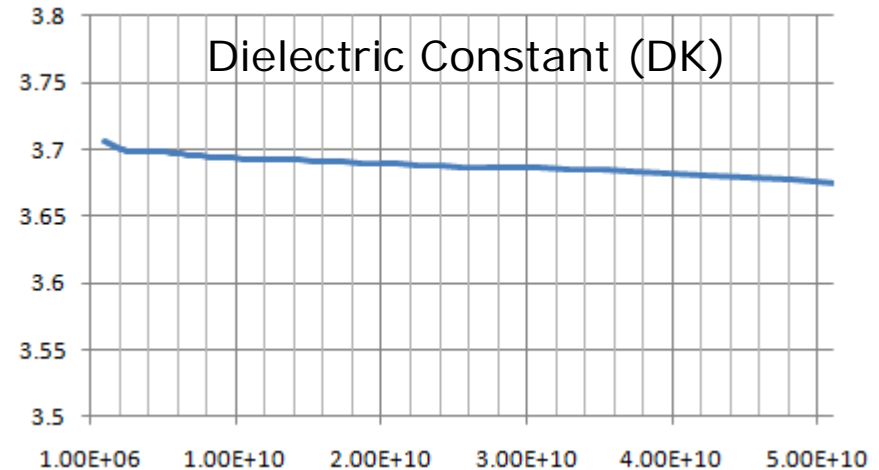
- 10-inch segment of micro-strip line
- PRBS7, 20 Gb/s, 10 ps rise time



Green – no roughness
Red – with roughness

Dielectric model for low loss material B

- Frequency-continuous 10-pole Debye model
- Very small changes in DK over the frequency range of interest
- LT is growing faster than allowed by wideband Debye model
- Roughness RMS=0.5 μm , Roughness Factor=2.5
- Suitable for applications up to 100 Gb/s



Conclusion

- The main result of this investigation is a simple and practical methodology to identify properties of dielectrics on the base of:
 - Precise measurement of generalized modal S-parameters of line segment
 - Accurate full-wave electromagnetic analysis with dispersive dielectric model and with all relevant loss and dispersion effects included
- Requires just 2 t-line segments and can be used on prototype and production boards for applications from 6-100 Gb/s
- Wideband Debye models can be effectively used for high-loss dielectrics and multi-pole Debye models for low-loss dielectrics

- Future work:
 - Automate the procedure for typical cases
 - Practical methodology to identify conductor roughness, effect of fibers,...

Be sure to visit us:

- Simberian Inc.
 - Booth #915 – Simbeor software and PLRD-1 Physical Layer Reference Design Board
 - www.simberian.com
- Teraspeed Consulting Group
 - www.teraspeed.com

Backup slides

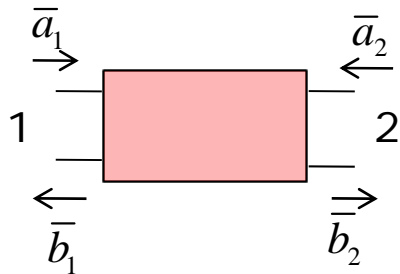
- Conversion of S-matrix to T-matrix

S-matrices and T-matrices

Arbitrary number of ports identical on the left and right side of multiport

$$\begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} = \begin{bmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{bmatrix} \cdot \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \end{bmatrix}$$

Cascading of 2 multiports described with S-parameters require solving a linear system



$$\begin{bmatrix} \bar{b}_1 \\ \bar{a}_1 \end{bmatrix} = \begin{bmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{bmatrix} \cdot \begin{bmatrix} \bar{a}_2 \\ \bar{b}_2 \end{bmatrix}$$

Cascading of 2 multiports described with T-parameters is simple product of two T-matrices

$$\begin{aligned} T_{1,1} &= S_{2,1} - S_{1,1} \cdot S_{2,1}^{-1} \cdot S_{2,2} \\ T_{1,2} &= S_{1,1} \cdot S_{2,1}^{-1} \\ T_{2,1} &= -S_{2,1}^{-1} \cdot S_{2,2} \\ T_{2,2} &= S_{2,1}^{-1} \end{aligned}$$

$$\begin{aligned} S_{1,1} &= T_{1,2} \cdot T_{2,2}^{-1} \\ S_{1,2} &= T_{1,1} - T_{1,2} \cdot T_{2,2}^{-1} \cdot T_{2,1} \\ S_{2,1} &= T_{2,2}^{-1} \\ S_{2,2} &= -T_{2,2}^{-1} \cdot T_{2,1} \end{aligned}$$

All elements are scalars in case of 2-ports (single-ended lines) or matrices in case of multi-conductor lines (differential)

See more in Carlin, Giordano, Network Theory, An Introduction to Reciprocal and Non-Reciprocal Circuits, 1964